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Laboratoire Ondes Milieux Complexes

Loughborough University

Acoustic Wave Propagation through a Random Dispersion of Solid Particles in a Viscous Fluid

PhD. Dissertation Defense

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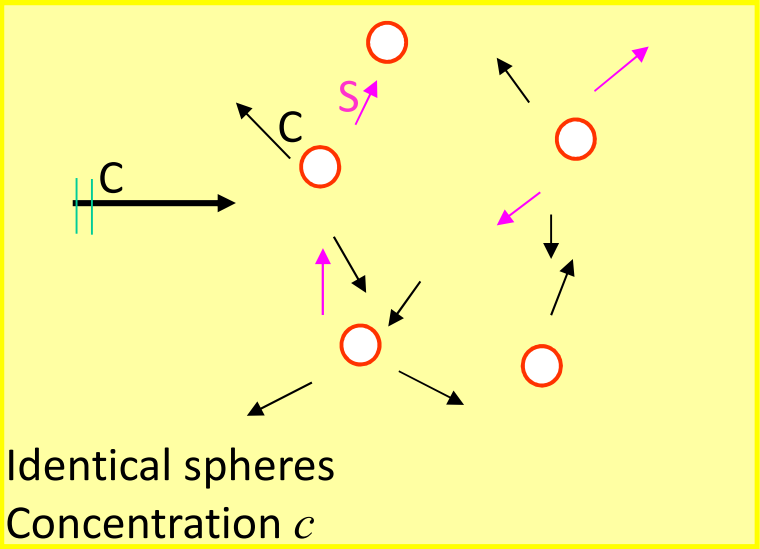
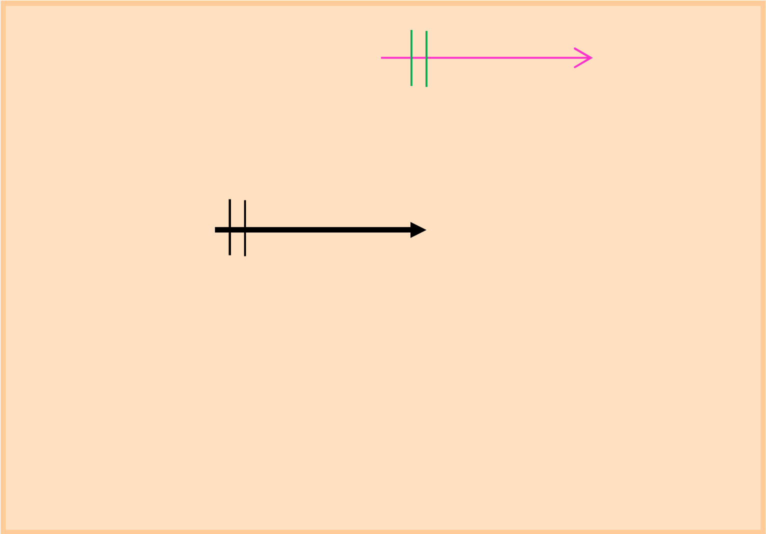
Motivation

Suspension of Solid Particles in a Viscous Liquid



Identical spheres
Concentration c

$\Leftrightarrow$



Long wavelength assumption

$\{\lambda, \mu, \rho\} \Leftrightarrow \{k_c, k_s, \rho\}$

Effective medium properties

Viscous liquid
$(\mu = -i\omega\eta)$
+
Solid particles

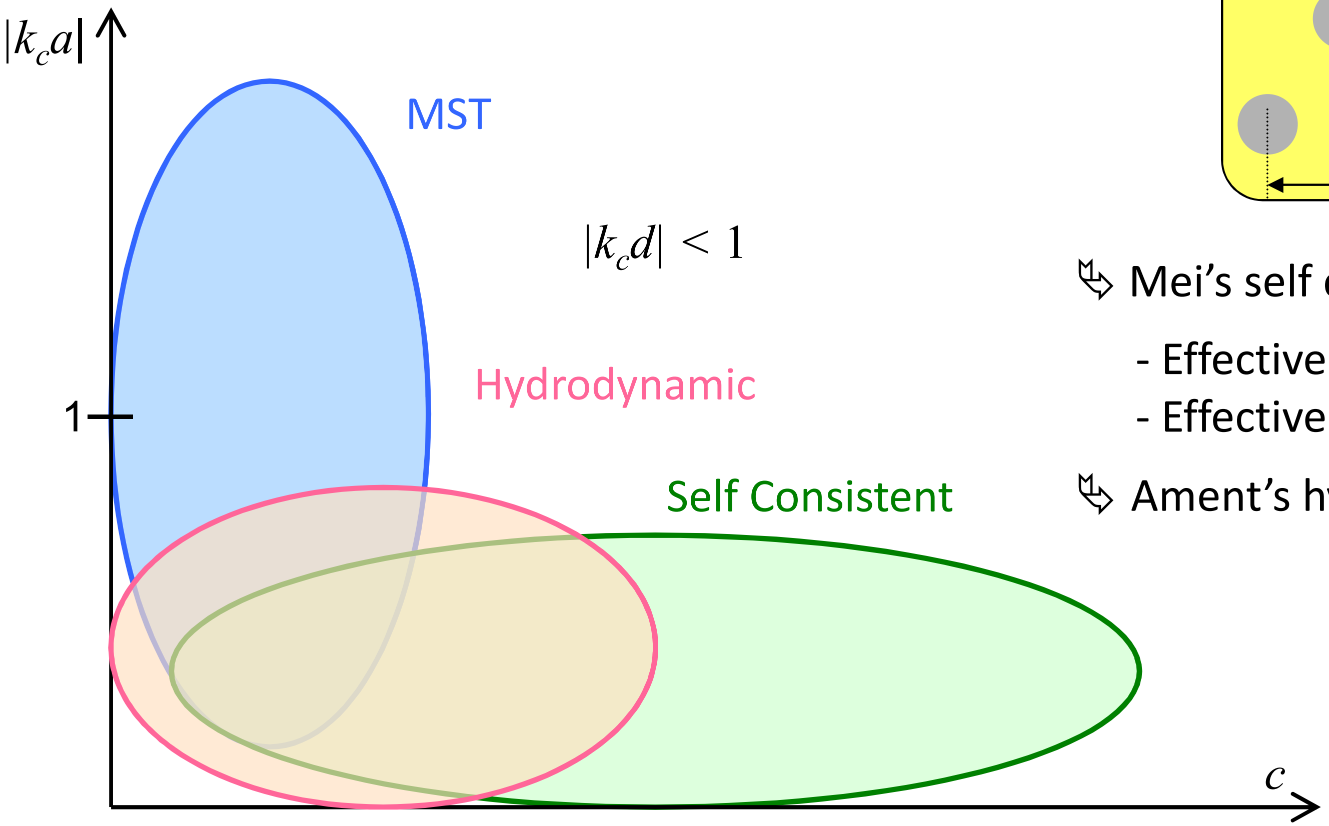
$$\begin{cases} B = \lambda + \dfrac{2}{3}\mu \\ \mu \\ \rho \end{cases}$$

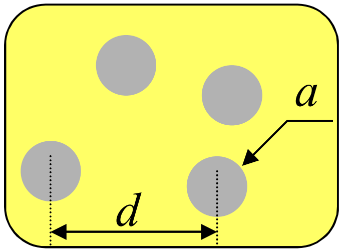
$$\begin{cases} B_{eff} = \lambda_{eff} + \dfrac{2}{3}\mu_{eff} \\ \mu_{eff} \\ \rho_{eff} \end{cases}$$

Effective properties depend on frequency, concentration, and scattering properties.

Outline

Modelling





- Mei’s self consistent model
  - Effective bulk modulus
  - Effective mass density
- Ament’s hydrodynamic model

[Mei, PRB, 2007]
[Ament, JASA, 1953]

Introduction

Objectives

To investigate effective properties of a viscous suspension:

- to obtain analytical expressions for scattering coefficients for a spherical particle for arbitrary partial wave order in visco-acoustic medium.
- to extend the self-consistent core-shell effective medium model through inclusion of the wave mode conversion and calculate effective dynamic properties.
- to investigate the dynamic effect of particle geometry on the effective mass density by extending a hydrodynamic model.

Outline

1. Motivation
2. Objectives
3. Single-Sphere Scattering
4. Effective Properties with Spheres [Mei, PRB, 2007]
5. Effective Properties with Spheroids [Ament, JASA, 1953]
6. Conclusion & Summary
7. Perspectives

Outline

1. Motivation
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Single-sphere Scattering

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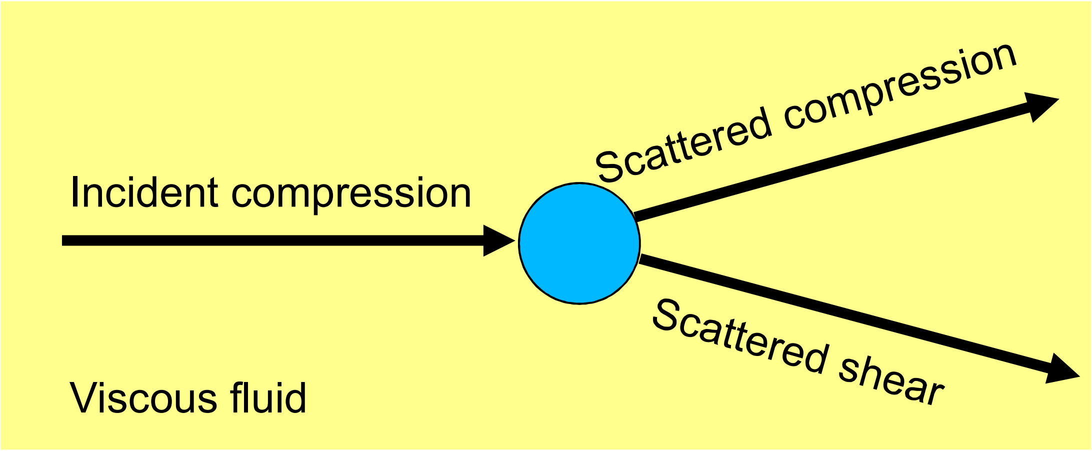
The Research Problem

Why single-particle scattering?

- **Single particle theory**
- **Multiple scattering theory**
- **Effective medium theory**

Wave Mode Conversion



Single-sphere scattering

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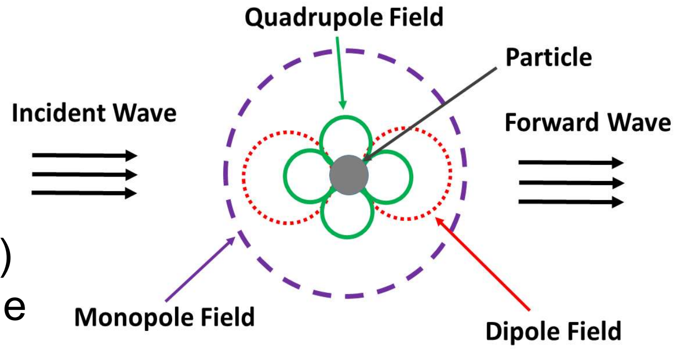
The Research Problem

Literature Review

- **Lord Rayleigh (1871)**
  - Rayleigh Scattering $|ka| \ll 1$



- **Allegra and Hawley (1972)** $|k_c a| \ll 1$
  - Particle: solid; Host: viscous fluid (compression)
  - Scattering coefficients order: monopole & dipole
- **Kuster and Toksoz (1974)**
  - Particle: solid; Host: solid
  - Rayleigh scattering $|k_c a| \ll 1, \quad |k_s a| \ll 1$
  - Scattering coefficients order: arbitrary
- **Pinfield and Forrester (2017)**
  - System: same as [A&H,1972] (shear)
  - Scattering coefficients order: dipole
- **This work** $|k_c a| \ll 1$
  - Scattering coefficients order: arbitrary

Single-sphere scattering

General background

Field equation for particle displacement, $\mathbf{u}$

$$(\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

Helmholtz decomposition

$$\mathbf{u}(\mathbf{r},t)=\boldsymbol{\nabla}\Phi(\mathbf{r},t)+\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r},t)$$

$$\boldsymbol{\nabla}\cdot\mathbf{A}=0$$

Scalar potential

Vector potential

Viscous fluid $\mu=-i\omega\eta$

$$k_c=\frac{\omega}{v_c}+i\alpha(\omega)$$

$$\mathrm{Re}(k_c)\gg\alpha(\omega)$$

$$|k_c|<|k_s|$$

$$k_s=\sqrt{\frac{\rho\omega}{2\eta}}(1+i)$$

Time harmonic potential $\exp(-i\omega t)$

$$(\nabla^2+k_c^2)\Phi=0 \qquad k_c^2=\frac{\rho}{\lambda+2\mu}\omega^2$$

$$(\nabla^2+k_s^2)\mathbf{A}=\mathbf{0} \qquad k_s^2=\frac{\rho}{\mu}\omega^2$$

$$|k_c|\sim|k_s|$$

Single-sphere scattering

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Solution of the Helmholtz equations

Spherical coordinate system (r, θ, ϕ)

$$r \in [0, \infty), \quad \theta \in [0, \pi], \; \phi \in [0, 2\pi)$$

$$\mathbf{A} = \boldsymbol{\nabla} \times (\mathbf{r}\psi) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\mathbf{r}\chi)$$

Debye potentials

$$\left(\nabla^2 + k_s^2\right)\mathbf{A} = \mathbf{0}$$

$$\left(\nabla^2 + k_s^2\right)(\psi, \chi) = 0$$

$$\begin{tikzcd} & \Phi \\ \Phi \arrow[ur] \arrow[dr] & \\ & \psi \end{tikzcd}$$

$$\begin{tikzcd} & \Phi \\ \psi \arrow[ur] \arrow[dr] & \\ & \psi \end{tikzcd}$$

$$\chi \rightarrow \chi$$

Plane compressional wave $\Phi = \Phi_0 \exp\left(i\mathbf{k}_c \cdot \mathbf{r}\right) = \Phi_0 \exp\left(i k_c r \cos\theta\right)$

$$\Phi_{inc} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_c r) P_n(\cos\theta)$$

Scattered waves

$$\Phi = \sum_{n=0}^{\infty} i^n (2n+1) T_n^{cc} h_n(k_c r) P_n(\cos\theta)$$

$$\psi = \sum_{n=0}^{\infty} i^n (2n+1) T_n^{cs} h_n(k_s r) P_n(\cos\theta)$$

Refracted waves

$$\Phi' = \sum_{n=0}^{\infty} i^n (2n+1) A_n^{cc} j_n(k'_c r) P_n(\cos\theta)$$

$$\psi' = \sum_{n=0}^{\infty} i^n (2n+1) A_n^{cs} j_n(k'_s r) P_n(\cos\theta)$$

Single-sphere scattering

Formulation

Displacement components

$$\mathbf{u}_C = \boldsymbol{\nabla}\Phi$$

$$\mathbf{u}_S = \boldsymbol{\nabla} \times \left(\mathbf{r}\psi\right)$$

Boundary condition at $r = a$

$$\begin{cases} u_r^+\left(r = a\right) = u_r^-\left(r = a\right) \\ u_\theta^+\left(r = a\right) = u_\theta^-\left(r = a\right) \\ \sigma_{rr}^+\left(r = a\right) = \sigma_{rr}^-\left(r = a\right) \\ \sigma_{r\theta}^+\left(r = a\right) = \sigma_{r\theta}^-\left(r = a\right) \end{cases}$$

Boundary equations

$$\mathbf{MX} = \mathbf{D}$$

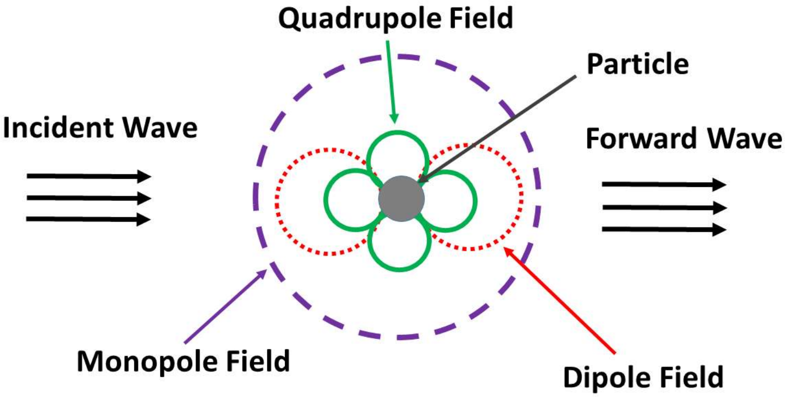
$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{44} & M_{44} \end{bmatrix} \begin{bmatrix} T_n^{cc} \\ T_n^{cs} \\ A_n^{cc} \\ A_n^{cs} \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix}$$

Single-sphere scattering

Scattering coefficients

Monopole n=0

$$T_0^{cc} = i\frac{(B-B')}{4\mu+3B'}y_c^3$$



Scattering profile of a single-sphere

$$|\,k_c a\,| \ll 1$$

Dipole n=1

$$\hat\rho = \frac{\rho'}{\rho}$$

$$y_c := k_c a$$
$$y_s := k_s a$$

$$T_1^{cc} = -\frac{i\left[h_1(y_s) - y_s h_1'(y_s)\right](\hat\rho-1)}{3\left[(4\hat\rho-7)h_1(y_s)+(2\hat\rho+1)y_s h_1'(y_s)\right]}y_c^3$$

$$T_1^{cs} = -\frac{(\hat\rho-1)}{\left[(4\hat\rho-7)h_1(y_s)+(2\hat\rho+1)y_s h_1'(y_s)\right]}y_c$$

$$T_1^{sc} = \frac{2(\hat\rho-1)}{y_s\left[(4\hat\rho-7)h_1(y_s)+(2\hat\rho+1)y_s h_1'(y_s)\right]}y_c^2$$

$$T_1^{ss} = -\frac{(4\hat\rho-7)j_1(y_s)+(2\hat\rho+1)y_s j_1'(y_s)}{\left[(4\hat\rho-7)h_1(y_s)+(2\hat\rho+1)y_s h_1'(y_s)\right]}$$

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Scattering coefficients

Quadrupole n=2

$$\begin{cases} T_2^{cc} = -\dfrac{2i\left[2h_2(y_s) - y_s h_2'(y_s)\right]}{135\left[3h_2(y_s) + y_s h_2'(y_s)\right]} y_c^5 \\[2ex] T_2^{cs} = -\dfrac{1}{9\left[3h_2(y_s) + y_s h_2'(y_s)\right]} y_c^2 \\[2ex] T_2^{sc} = -\dfrac{2}{3y_s\left[3h_2(y_s) + y_s h_2'(y_s)\right]} y_c^3 \\[2ex] T_2^{ss} = -\dfrac{3j_2(y_s) + y_s j_2'(y_s)}{\left[3h_2(y_s) + y_s h_2'(y_s)\right]} \end{cases}$$

n Poles

$$\begin{cases} T_n^{cc} = O\left(y_c^{2n+1}\right) \\ T_n^{cs} = O\left(y_c^{n}\right) \\ T_n^{sc} = O\left(y_c^{n+1}\right) \\ T_n^{ss} = O\left(y_c^{0}\right) \end{cases}$$

Single-sphere scattering

Numerical results

Normalised scattering coefficients

$$y_c := k_c a \qquad y_s := k_s a$$

$$\begin{cases} T^{cc}_{nN} = \dfrac{T^{cc}_n}{y_c^{2n+1}} \\[2ex] T^{cs}_{nN} = \dfrac{T^{cs}_n\, h_n\left(y_s\right)}{y_c^n} \\[2ex] T^{sc}_{nN} = \dfrac{T^{sc}_n}{j_n\left(y_s\right)\, y_c^{n+1}} \\[2ex] T^{ss}_{nN} = \dfrac{T^{ss}_n\, h_n\left(y_s\right)}{j_n\left(y_s\right)} \end{cases}$$

Physical properties of silica and water

Parameters	**Water**	**Silica**
Sound speed ($m.s^{-1}$)	1497	5968
Density ($kg.m^{-3}$)	997	2100
Shear viscosity (Pa.s)	8.91×10^{-4}	0
Shear modulus (GPa)	0	30.9
Attenuation factor ($Np.m^{-1}.MHz^{-2}$)	2.3×10^{-2}	2.6×10^{-10}

Frequency range 0.01 – 10 MHz

$$\boxed{\left|k_c a\right| \ll 1} \qquad \boxed{\Re\mathrm{e}(k_s a) \in \left[10^{-2},\, 10^{2}\right]}$$

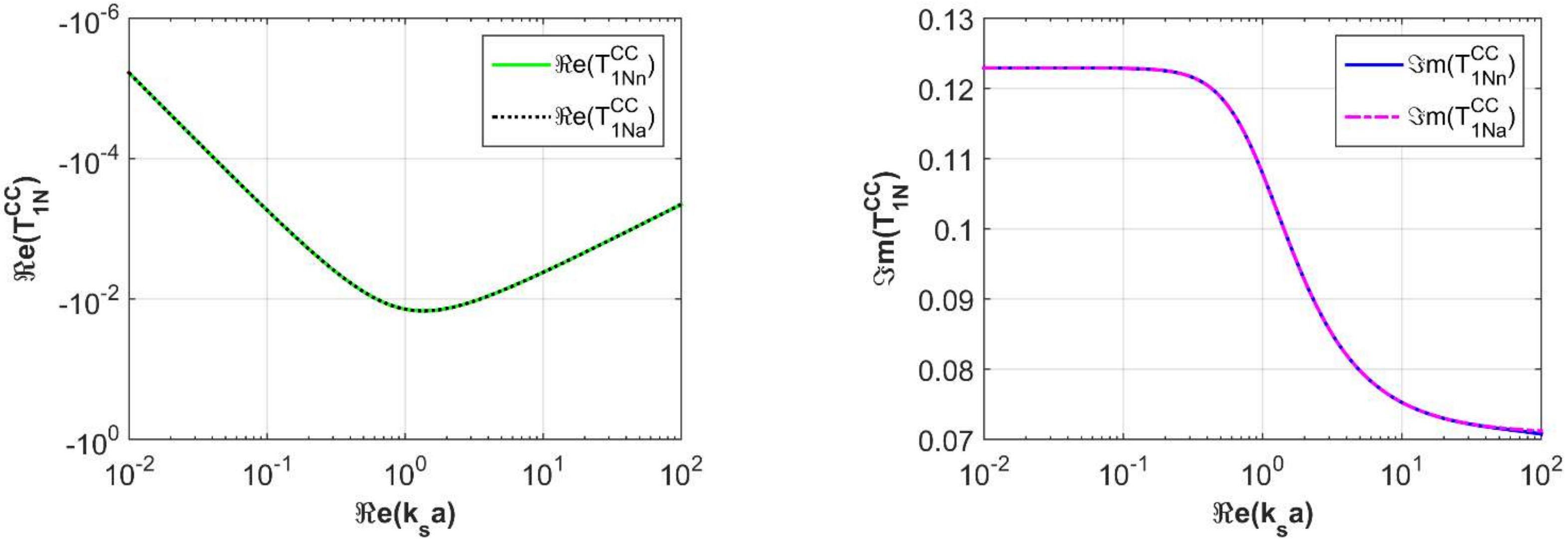
$$\boxed{a = 300\ \mu m}$$

$$\boxed{\Re\mathrm{e}(k_s a) = 100 \longrightarrow \Re\mathrm{e}(k_c a) = 0.04}$$

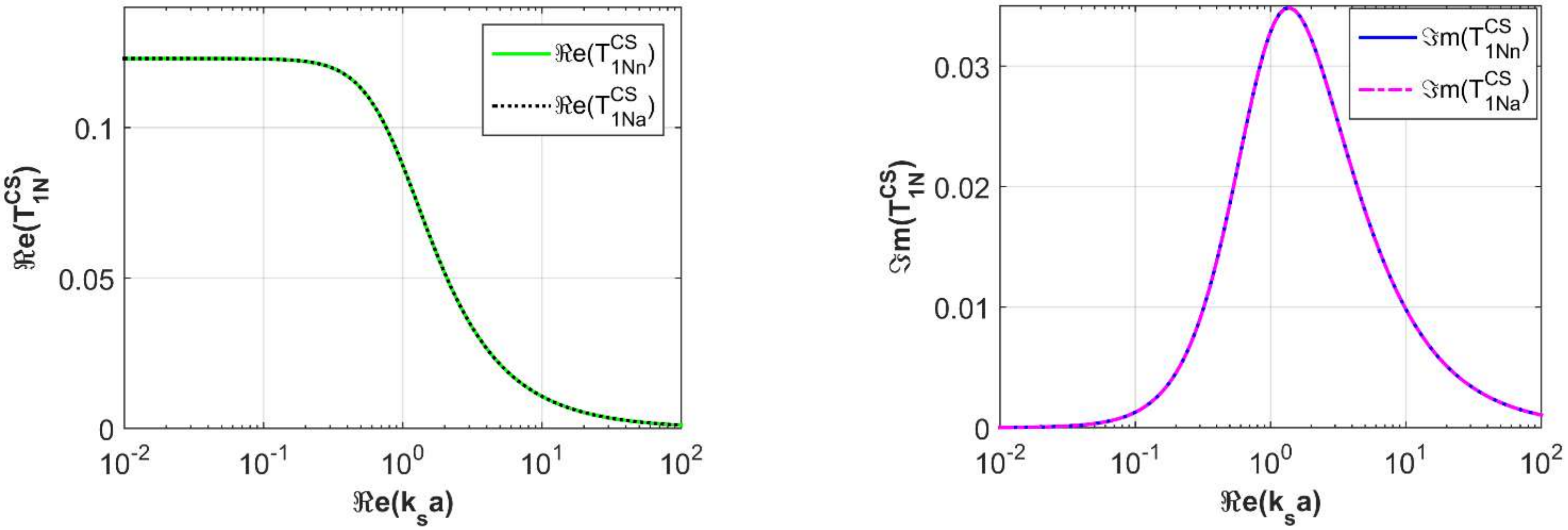
Single-sphere scattering

Numerical results: Dipole coefficients

T_{1N}^{cc}



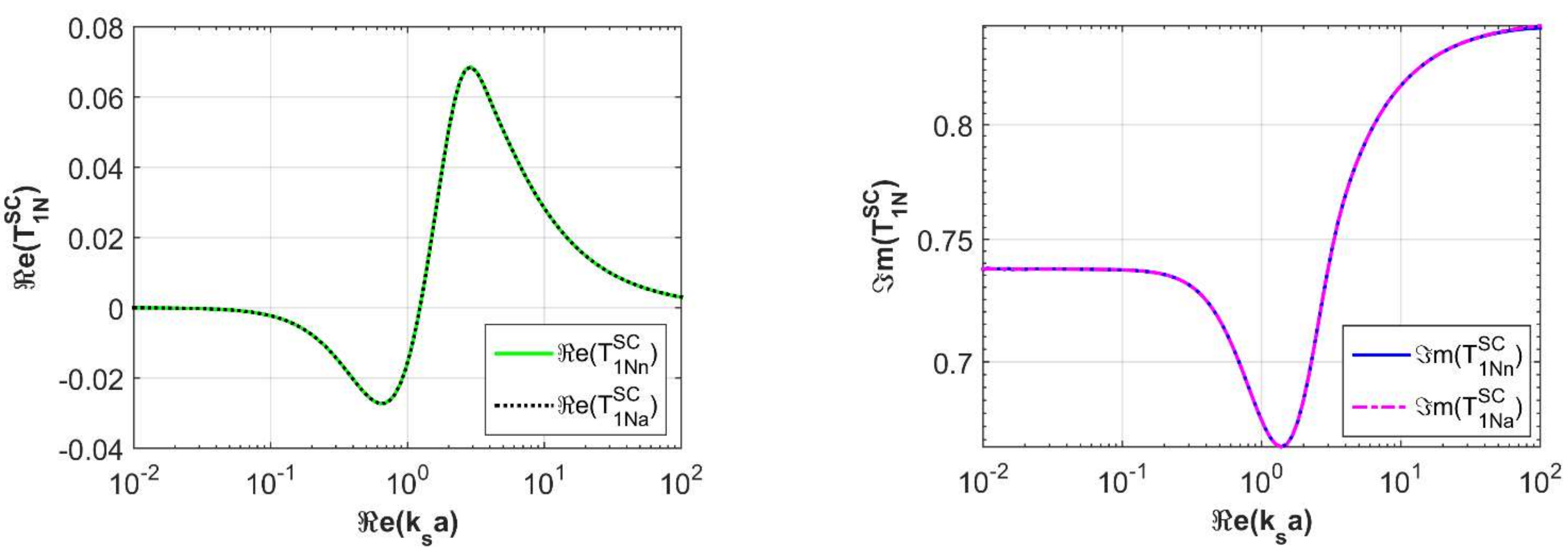
T_{1N}^{cs}



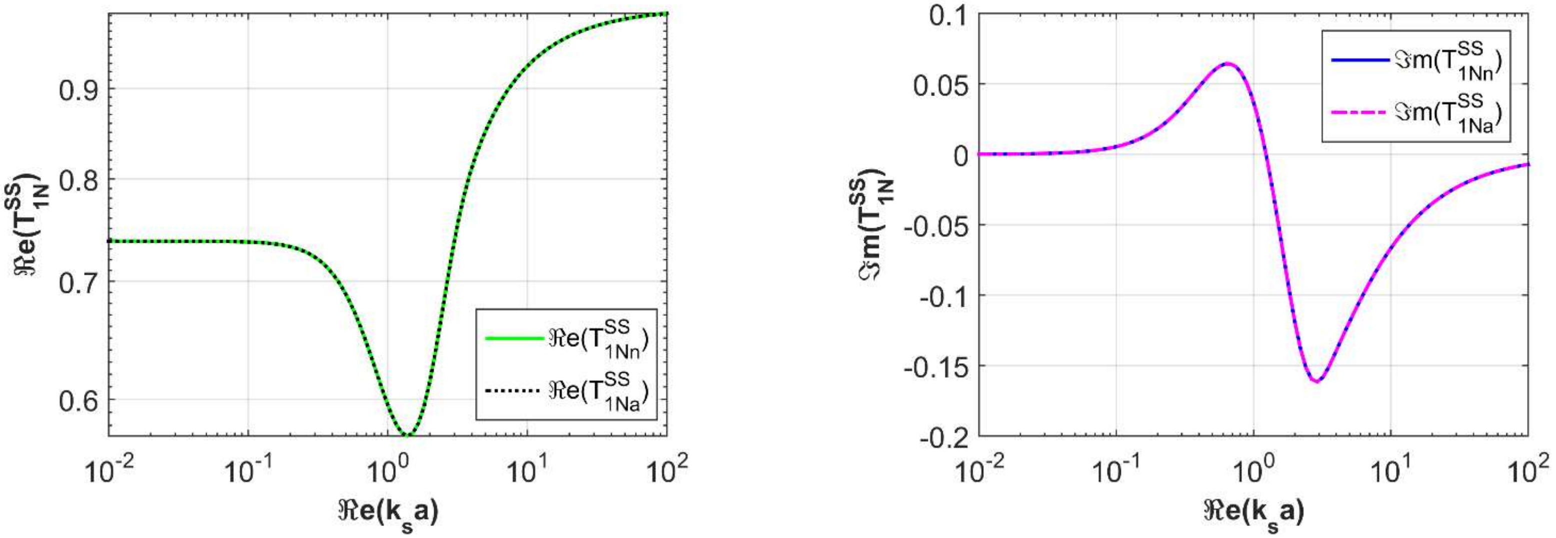
Single-sphere scattering

Numerical results: Dipole coefficients

T^{SC}_{1N}



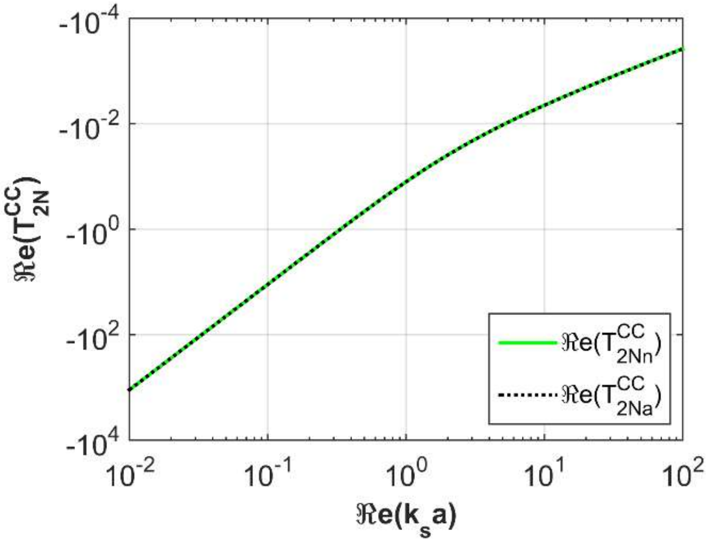
T^{SS}_{1N}

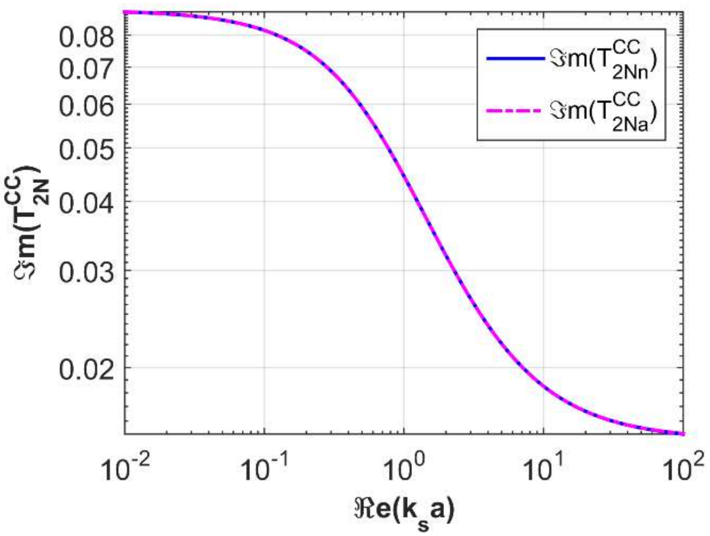


Single-sphere scattering

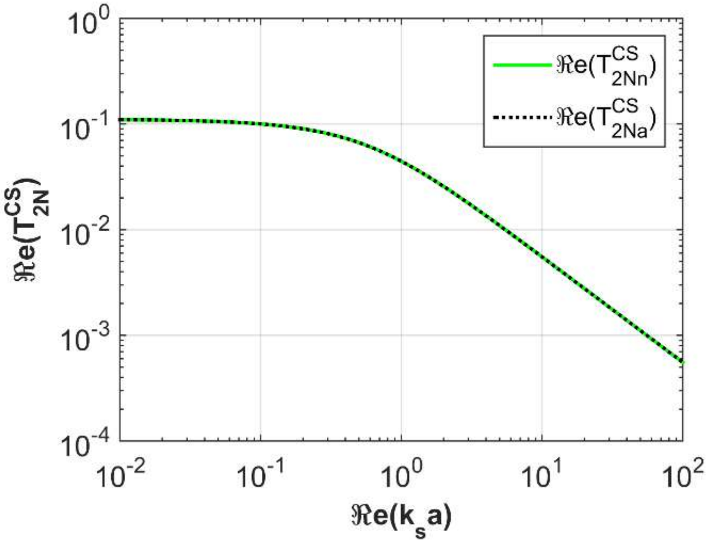
Numerical results: Quadrupole coefficients

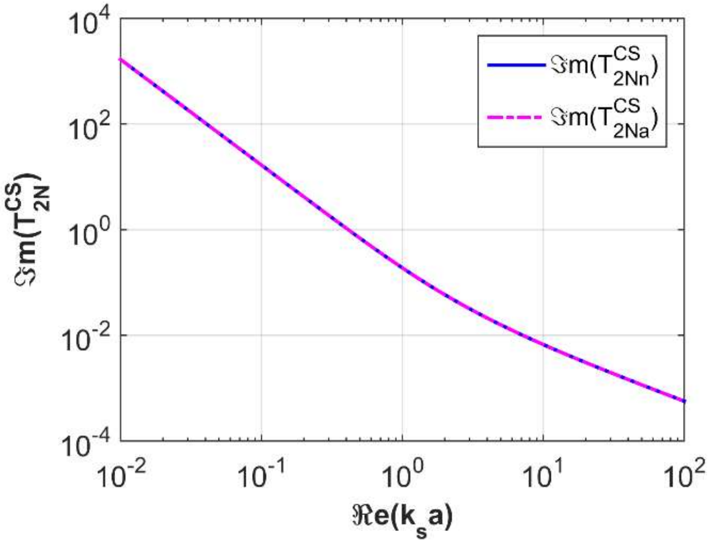
T^{cc}_{2N}





T^{cs}_{2N}

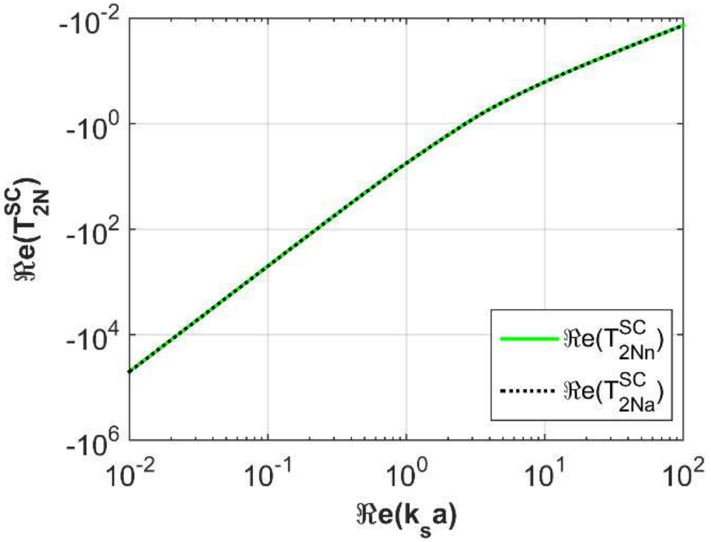


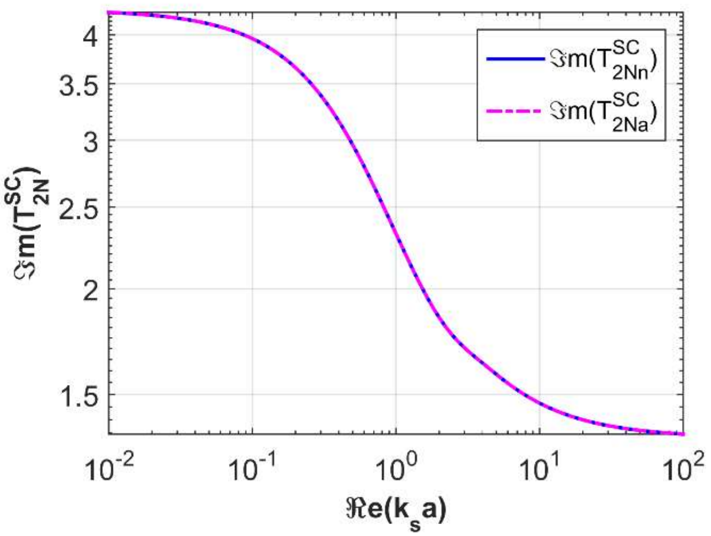


Single-sphere scattering

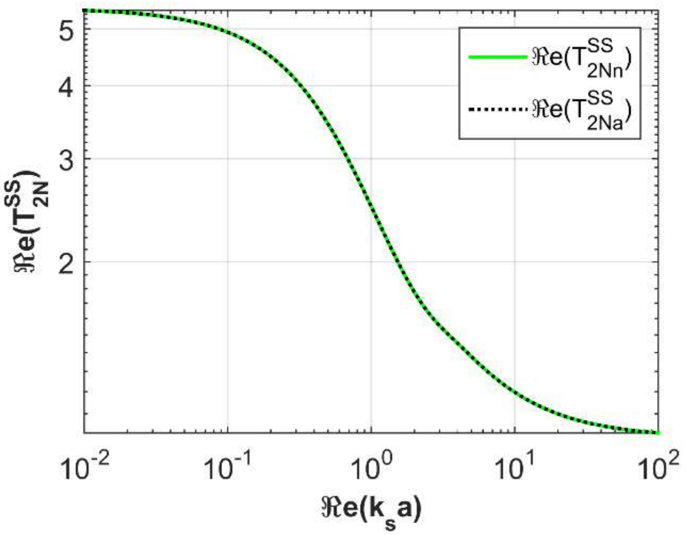
Numerical results: Quadrupole coefficients

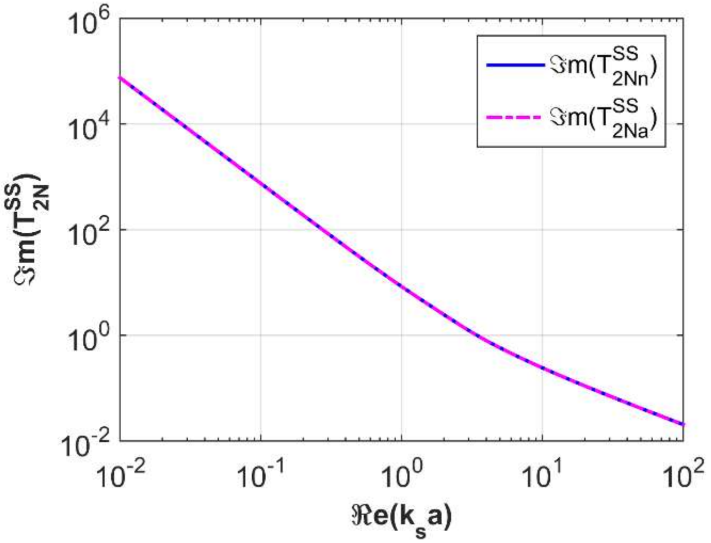
$\mathrm{T}^{\mathrm{SC}}_{2\mathrm{N}}$





$\mathrm{T}^{\mathrm{SS}}_{2\mathrm{N}}$





Single-sphere scattering

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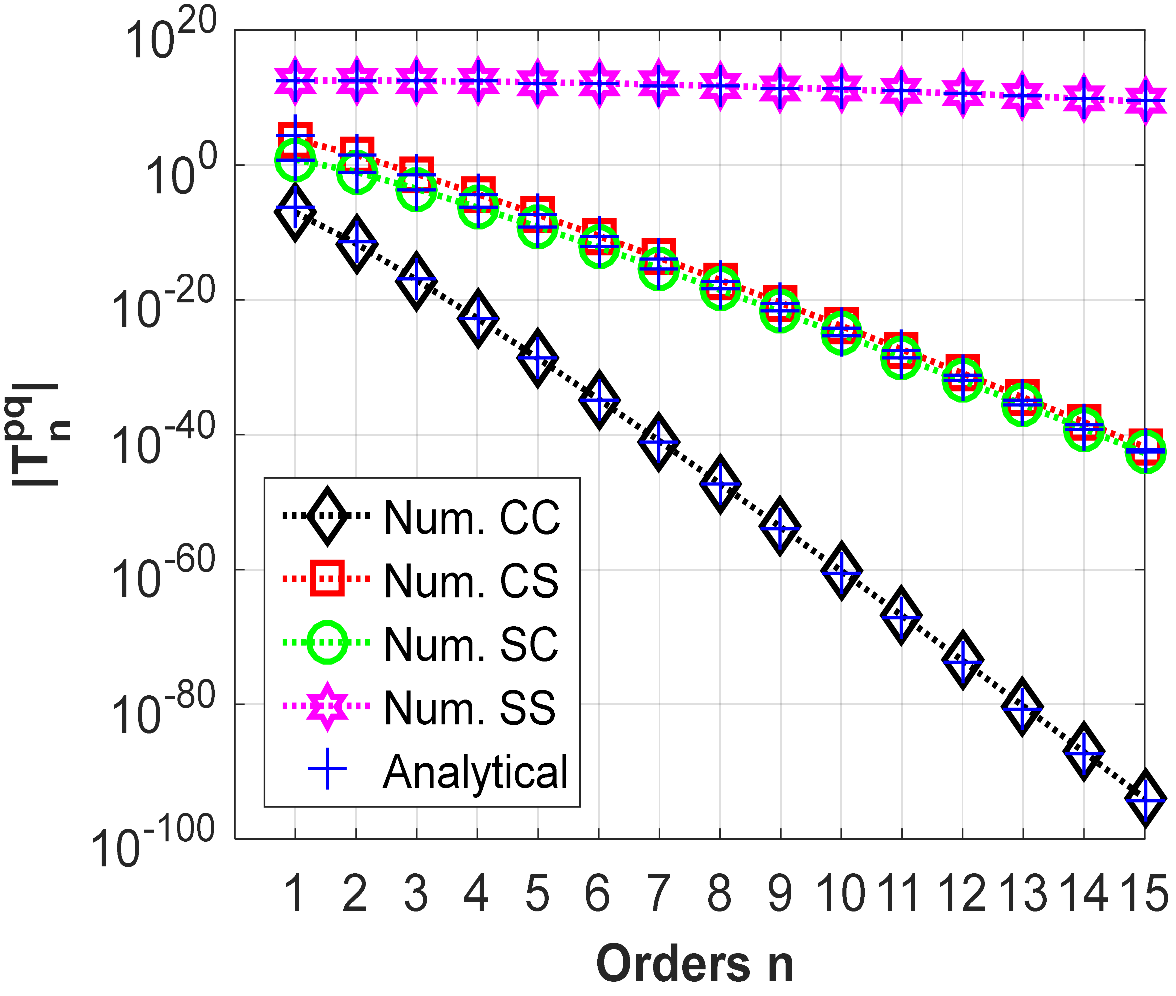
Numerical results: general results

$$y_c := k_c a$$
$$y_s := k_s a$$

$$y_c < y_s$$



Summary

- Analytical approximations have been derived for scattering coefficients for general partial wave orders for a sphere in a viscous liquid.
- Good agreement between the analytical and numerical solutions within the validity region.

Outline

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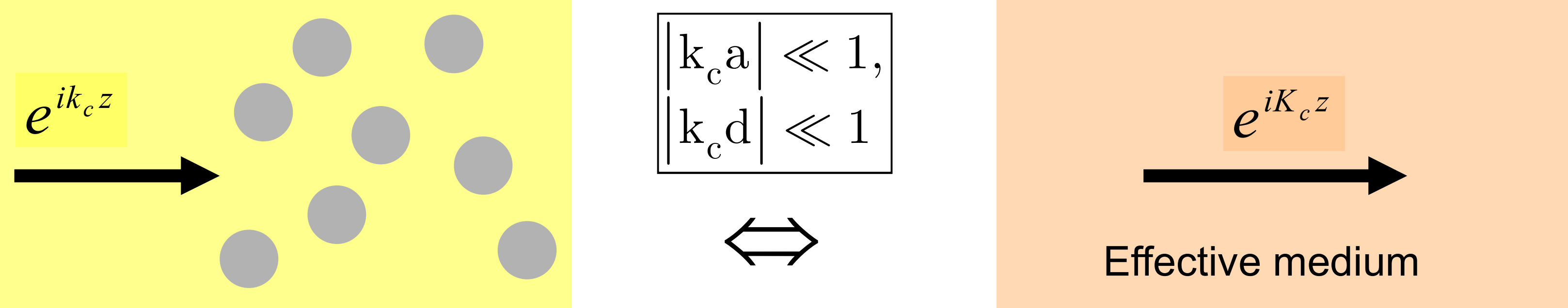
1. Motivation
2. Objectives
3. Single-Sphere Scattering
4. **Effective Properties with Spheres**
5. Effective Properties with Spheroids
6. Conclusion & Summary
7. Perspectives

Effective density with spheres

Effective medium



$$\left|k_c a\right| \ll 1, \quad \left|k_c d\right| \ll 1$$

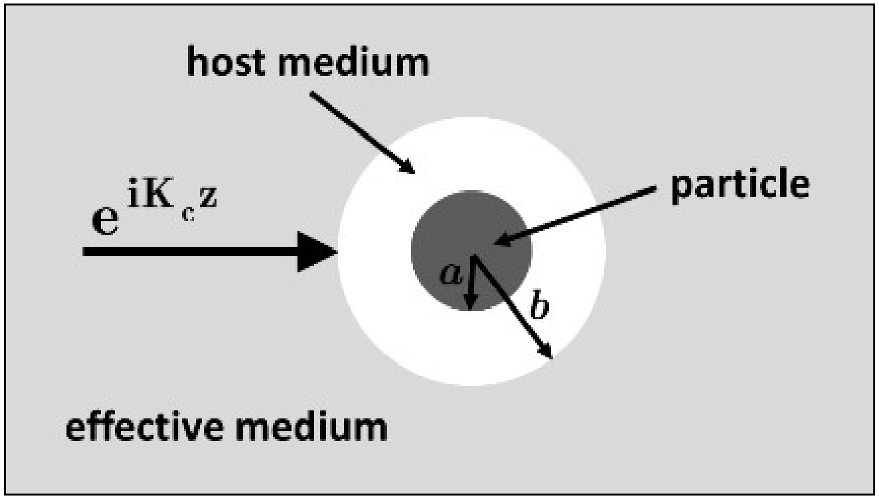
$$\Leftrightarrow$$

- Core-shell model:
  - Concentration:

$$c = \frac{a^3}{b^3}$$

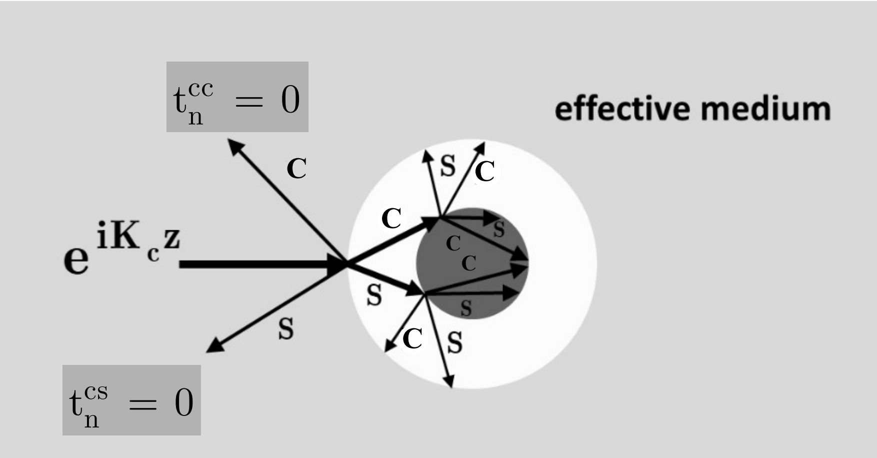
  - Self-consistency:

$$t_n^{pq} = 0$$



Effective density with spheres

Method



Boundary condition at $r = b$

$$\begin{cases} u_r^+(r=b) = u_r^-(r=b) \\ u_\theta^+(r=b) = u_\theta^-(r=b) \\ \sigma_{rr}^+(r=b) = \sigma_{rr}^-(r=b) \\ \sigma_{r\theta}^+(r=b) = \sigma_{r\theta}^-(r=b) \end{cases}$$

Boundary equations

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{44} & M_{44} \end{bmatrix} \begin{bmatrix} t_n^{cc} \\ t_n^{cs} \\ C_n^{cc} \\ C_n^{cs} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$

Effective density with spheres

Effective bulk modulus

Monopole mode n = 0
No shear wave
Viscosity in complex λ and μ

$$X_c := K_c b \qquad x_c := k_c b$$

$$|X_c| \ll 1, \qquad |x_c| \ll 1$$

Polar displacement and stress boundary equations vanish

Solve radial displacement and stress equations

Self-consistency $t_0^{cc} = 0$

$$B_{\mathrm{eff}} = \frac{B\left(k_c b\right)^3 + 4i\mu T_0^{cc}}{\left(k_c b\right)^3 - 3iT_0^{cc}}$$

Quasi-static

$$T_0^{cc} \propto \left(k_c b\right)^3$$

Effective density with spheres

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Effective bulk modulus

Quasi-static

$$B_{\text{eff}} = \frac{B\left(k_c b\right)^3 + 4i\mu T_0^{cc}}{\left(k_c b\right)^3 - 3iT_0^{cc}}$$

$$T_0^{cc} = ic\frac{\left(B - B'\right)}{4\mu + 3B'}\left(k_c b\right)^3$$

Kuster-Toksoz

$$\frac{B - B_{\text{eff}}}{3B_{\text{eff}} + 4\mu} = c\frac{B - B'}{3B' + 4\mu}$$

Solid in solid

Inviscid host $\mu \rightarrow 0$

$$\frac{1}{B_{\text{eff}}} = \frac{(1-c)}{B} + \frac{c}{B'}$$

Harmonic average

Compressibility $\kappa = 1 / B$

$$\kappa_{\text{eff}} = (1-c)\kappa + c\kappa'$$

Volume-averaged

[Kuster, Geophys, 2002]

Effective density with spheres

Effective dynamic mass density

Dipole mode n=1

- Rearrangement of boundary equations
- Leading-order in $\mathrm{X_c}$ and $\mathrm{x_c}$

$$\mathrm{X_c} := \mathrm{K_c b}$$
$$\mathrm{x_c} := \mathrm{k_c b}$$

$$|\mathrm{X_c}| \ll 1,$$
$$|\mathrm{x_c}| \ll 1$$

$$\frac{\rho_{\mathrm{eff}}}{\rho} = \frac{\mathrm{E}}{\mathrm{F}}$$

$$\mathrm{E} = 3\left(\mathrm{T}_1^{cs}\mathrm{T}_1^{sc} - \mathrm{T}_1^{cc}\mathrm{T}_1^{ss}\right)\left[\mathrm{x_s\, h}_1'(\mathrm{x_s}) - 7\mathrm{h}_1(\mathrm{x_s})\right] - 3\mathrm{T}_1^{cc}\left[\mathrm{x_s\, j}_1'(\mathrm{x_s}) - 7\mathrm{j}_1(\mathrm{x_s})\right] - 3\mathrm{x_c}\mathrm{T}_1^{sc} - 6\mathrm{T}_1^{cs}\frac{\mathrm{x_c^2}}{\mathrm{x_s}}$$
$$+ i\mathrm{x_c^3}\mathrm{T}_1^{ss}\left[\mathrm{h}_1(\mathrm{x_s}) - \mathrm{x_s\, h}_1'(\mathrm{x_s})\right] + i\mathrm{x_c^3}\left[\mathrm{j}_1(\mathrm{x_s}) - \mathrm{x_s\, j}_1'(\mathrm{x_s})\right]$$

$$\mathrm{F} = 6\left(\mathrm{T}_1^{cc}\mathrm{T}_1^{ss} - \mathrm{T}_1^{cs}\mathrm{T}_1^{sc}\right)\left[\mathrm{x_s\, h}_1'(\mathrm{x_s}) + 2\mathrm{h}_1(\mathrm{x_s})\right] + 6\mathrm{T}_1^{cc}\left[\mathrm{x_s\, j}_1'(\mathrm{x_s}) + 2\mathrm{j}_1(\mathrm{x_s})\right] - 3\mathrm{x_c}\mathrm{T}_1^{sc} - 6\mathrm{T}_1^{cs}\frac{\mathrm{x_c^2}}{\mathrm{x_s}}$$
$$+ i\mathrm{x_c^3}\mathrm{T}_1^{ss}\left[\mathrm{h}_1(\mathrm{x_s}) - \mathrm{x_s\, h}_1'(\mathrm{x_s})\right] + i\mathrm{x_c^3}\left[\mathrm{j}_1(\mathrm{x_s}) - \mathrm{x_s\, j}_1'(\mathrm{x_s})\right]$$

Effective density with spheres

Effective dynamic mass density: low $|k_s a|$ form

$$y_c := k_c a$$
$$y_s := k_s a$$

$$|x_c| := |k_c b| \ll 1$$
$$|x_s| := |k_s b| \ll 1$$

$$a = c^{\frac{1}{3}} b$$

$$x_c := c^{\frac{1}{3}} y_c$$
$$x_s := c^{\frac{1}{3}} y_s$$

Low $|x_s|$ form of coefficients

$$T_1^{cc} = \frac{1}{9} i c y_c^3 \left(\hat{\rho} - 1\right) \qquad T_1^{cs} = \frac{1}{9} i c y_s^2 y_c \left(\hat{\rho} - 1\right)$$

$$T_1^{sc} = \frac{2}{9} i c y_s y_c^2 \left(\hat{\rho} - 1\right) \qquad T_1^{ss} = \frac{2}{9} i c y_s^3 \left(\hat{\rho} - 1\right)$$

$$\frac{\rho_{\mathrm{effL}}}{\rho}\left(x_s\right) = \left(1 - c\right) + c\hat{\rho} + J\left(c\right) x_s^2 + K\left(c\right) x_s^4 + O\left(x_s^6\right)$$

Static limit

$$\lim_{\substack{x_c \to 0 \\ x_s \to 0}} \frac{\rho_{\mathrm{eff}}}{\rho} = \left(1 - c\right) + c\hat{\rho}$$

Volume average

Agreement with the Ament’s static limit

[Ament, JASA, 2005]

Effective density with spheres

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Effective mass density : high $|k_s a|$

Asymptotic form of scattering coefficients

$$\begin{cases} T_1^{cc} \sim \dfrac{i}{3}\dfrac{\hat{\rho}-1}{2\hat{\rho}+1}y_c^3 \\ T_1^{cs} \sim -ie^{-iy_s}\dfrac{\hat{\rho}-1}{2\hat{\rho}+1}y_c \\ T_1^{sc} \sim -2i\dfrac{e^{-iy_s}}{y_s}\dfrac{\hat{\rho}-1}{2\hat{\rho}+1}y_c^2 \\ T_1^{ss} \sim -i\,e^{-iy_s}\sin\left(y_s\right) \end{cases}$$

$$k_s a = \left(1+i\right)\beta a \qquad \left|k_s a\right| \gg 1$$

$$\beta = \sqrt{\rho\omega / (2\eta_s)} \gg 1$$

$$\frac{\rho_{eff}}{\rho} \sim 1 - 9ic\frac{T_1^{cc}}{y_c^3 + 6icT_1^{cc}}$$

Inviscid form

$$\frac{\rho_{eff}}{\rho} = \frac{\left(\rho+2\rho'\right)+c\left(\rho'-\rho\right)}{\left(\rho+2\rho'\right)-2c\left(\rho'-\rho\right)}$$

Ament's formula

[Ament, JASA, 2005]

Effective density with spheres

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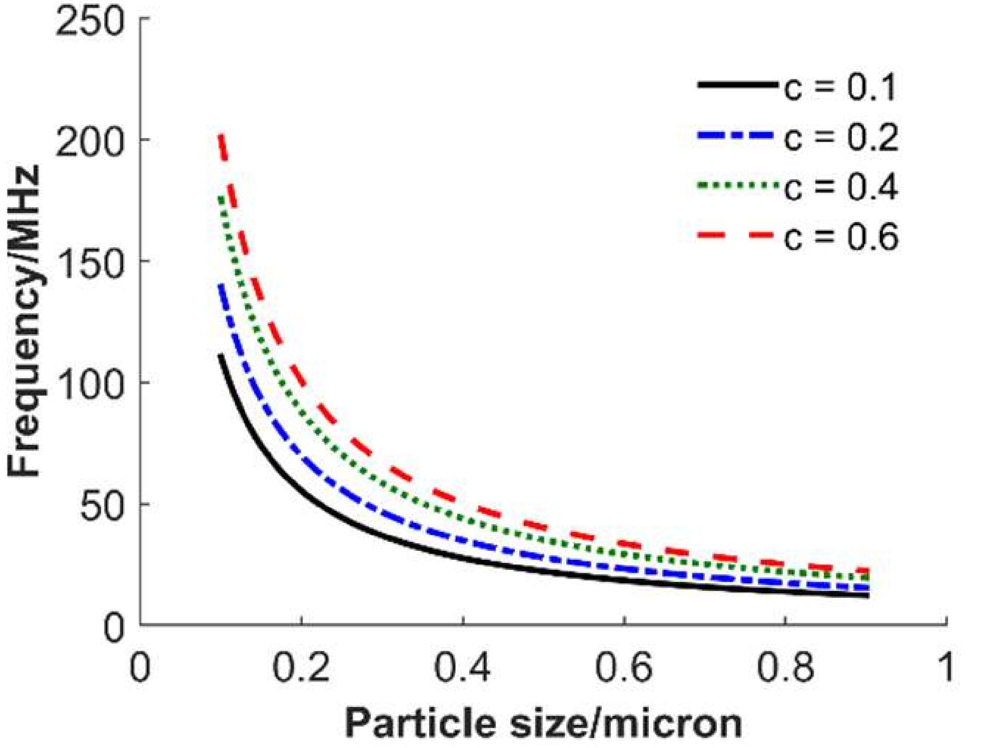
Numerical results

Validity domain of the model

$$\boxed{\left|k_c b\right| \ll 1}$$

$$\boxed{\left|k_c b\right| \leq \left|k_c b\right|_{\max}} \quad \Rightarrow a\ f < \left|k_c b\right|_{\max} \frac{v_c}{2\pi} c^{1/3} \quad \boxed{\left|k_c b\right|_{\max} = 0.1}$$

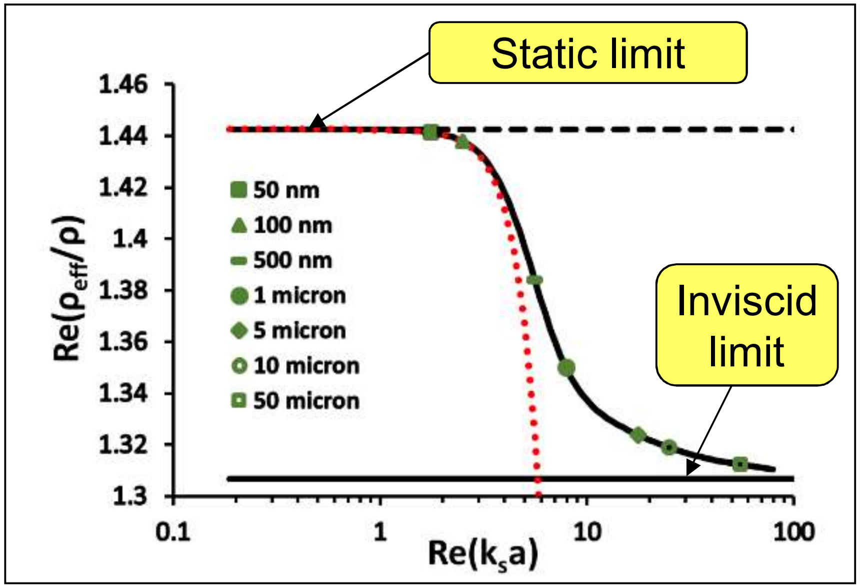


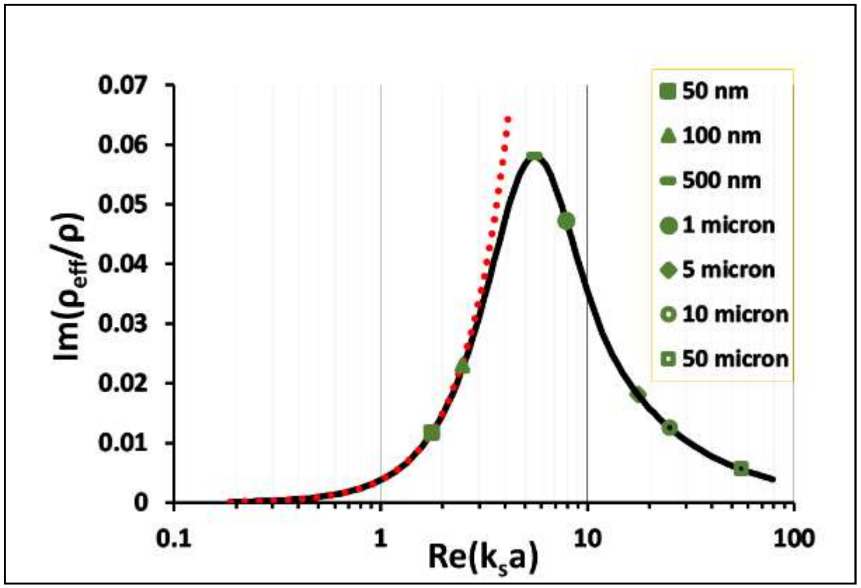
$$\boxed{\left[\Re e\left(k_s a\right)\right]^2_{\max} = \frac{1}{2}\left|k_c b\right|_{\max} \frac{\left|k_s^2\right|}{\left|k_c\right|} c^{1/3} a_0}$$

Effective density with spheres

Numerical results: Variation with frequency at different particle sizes

Resonance-like behaviour like MST

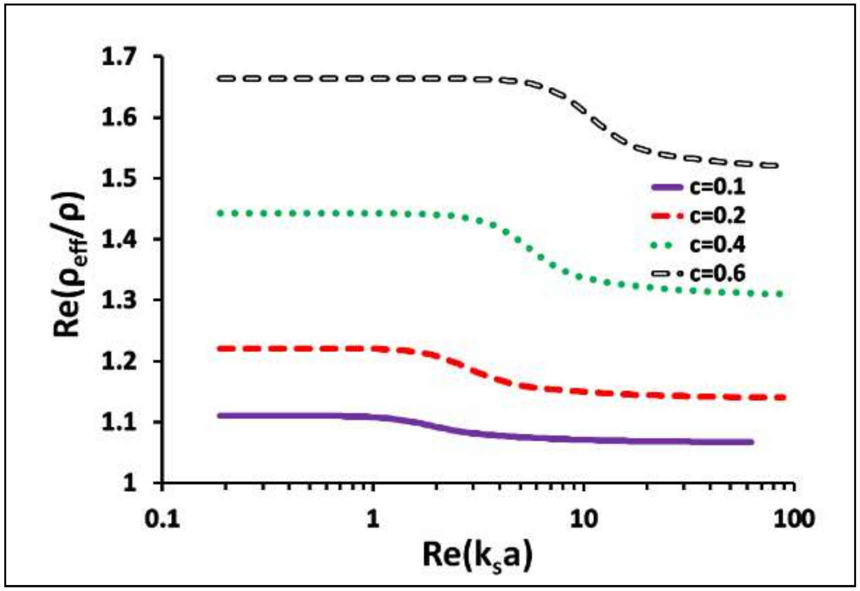


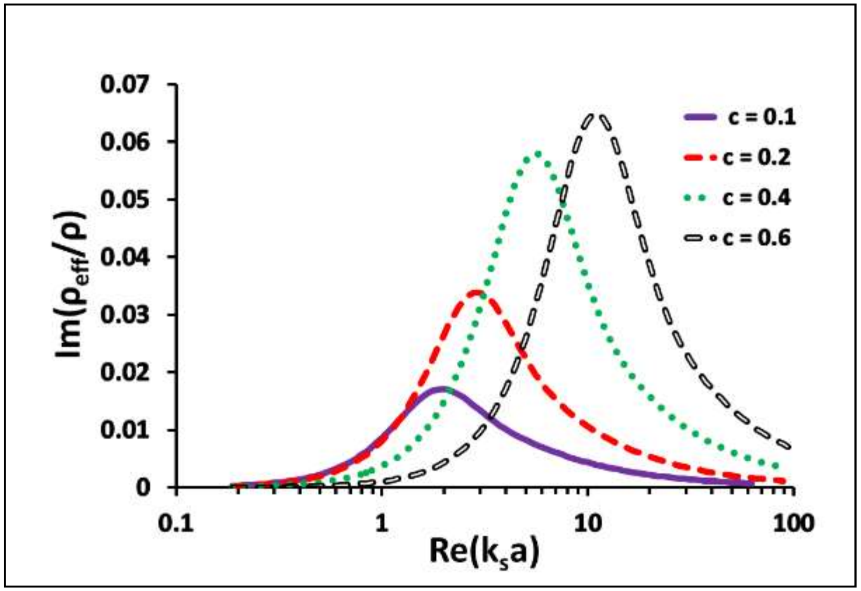


$$a = 100\ \mu m$$
$$c = 0.4$$

Effective density with spheres

Numerical results: Variation with frequency at various concentrations





$$a = 100\ \mu m$$
$$c = 0.1 \text{ to } 0.6$$

Effective density with spheres

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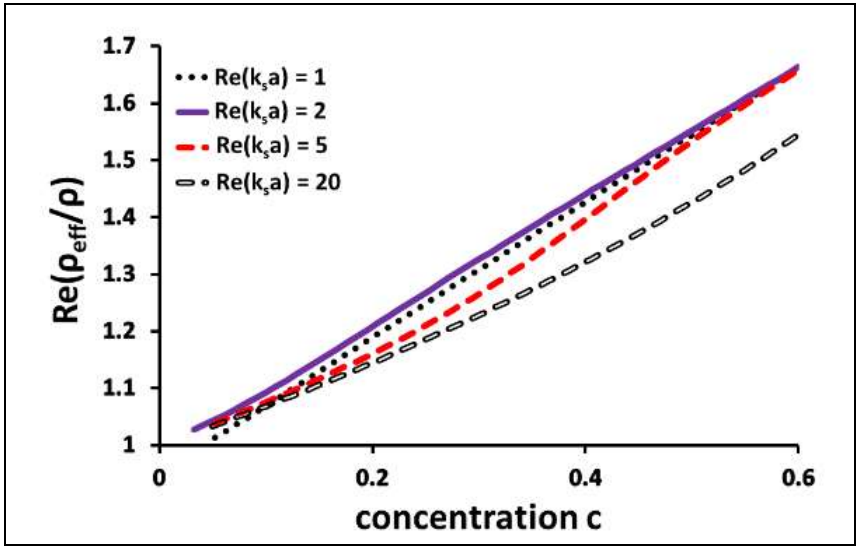
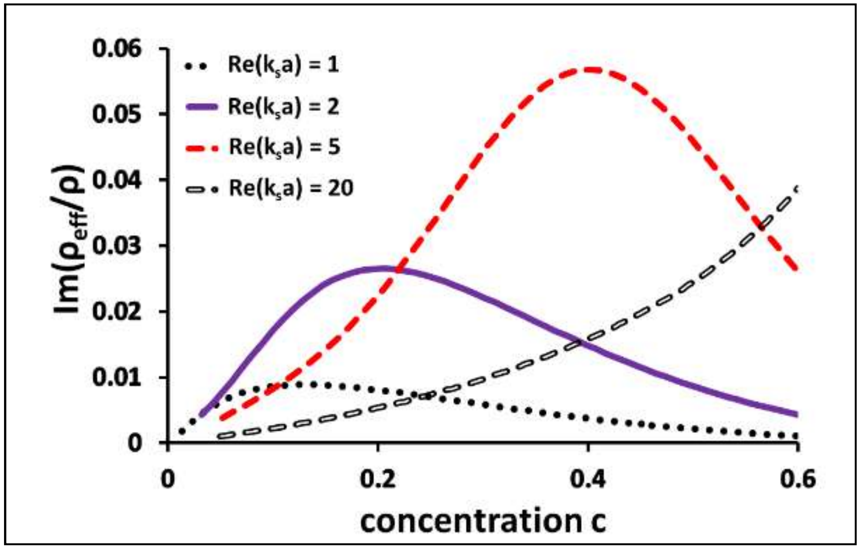
Numerical results: variation with concentration at fixed $\mathrm{Re}(k_s a)$

$\mathrm{Re}(k_s a)$	**a (nm)**	**f (MHz)**	**c_{min} (%v/v)**
1	50	113.9	1.4
2	150	50.6	3.2
5	800	11.1	5.2
20	13,000	0.67	5.0

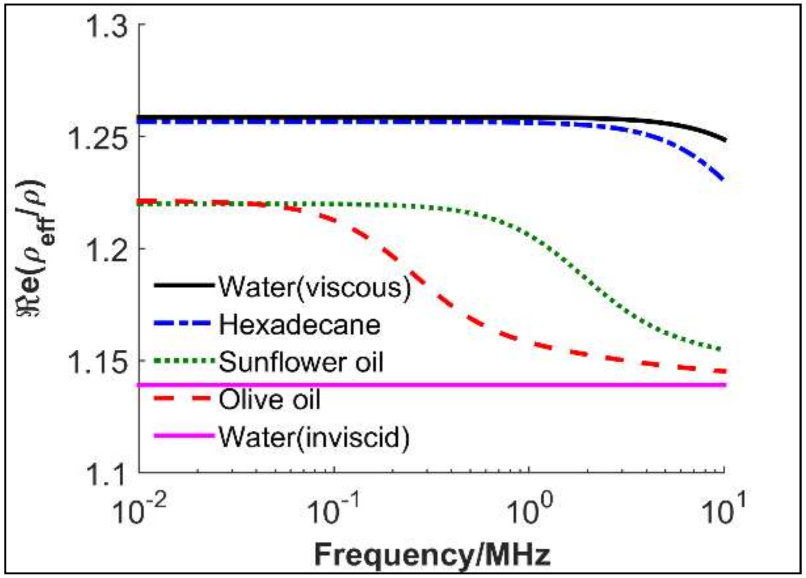




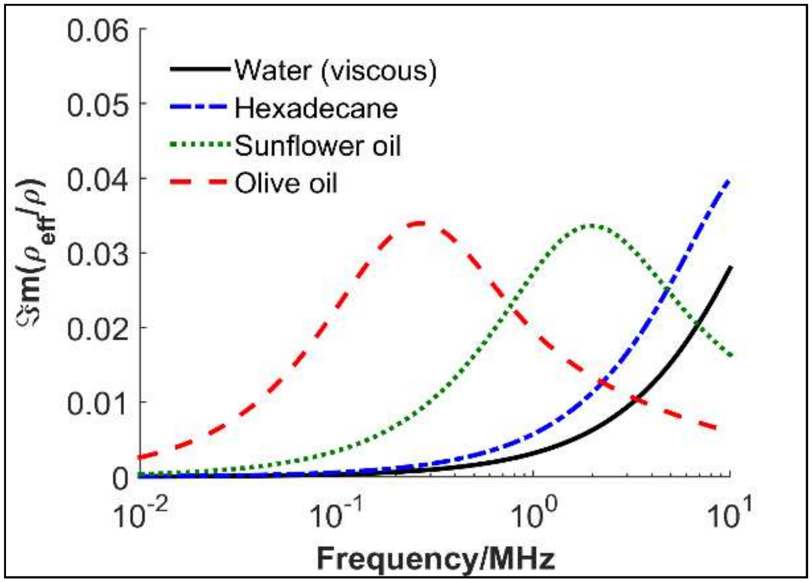
Effective density with spheres

Numerical results:

Host fluid	Water	Hexadecane	Sunflower oil	Olive oil
Sound velocity ($m.s^{-1}$)	1497	1299	1470	1464
Density ($kg.m^{-3}$)	997	1000	920.6	915.8
Shear viscosity (Pa.s)	0.000891	0.00663	0.054	0.092
Attenuation coefficient ($Np.m^{-1}.MHz^{-p}$)	0.023	0.145	1.15	1.625
Attenuation exponent p	2	2	1.77	2



$$a = 3\ \mu m$$
$$c = 0.25$$



Conclusion

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- Extension of a core-shell effective medium model by taking the viscosity of the host fluid into account through wave mode conversion
- Calculation of the effective bulk modulus and effective density

Outline

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Laboratoire Ondes Milieux Complexes

1. Motivation
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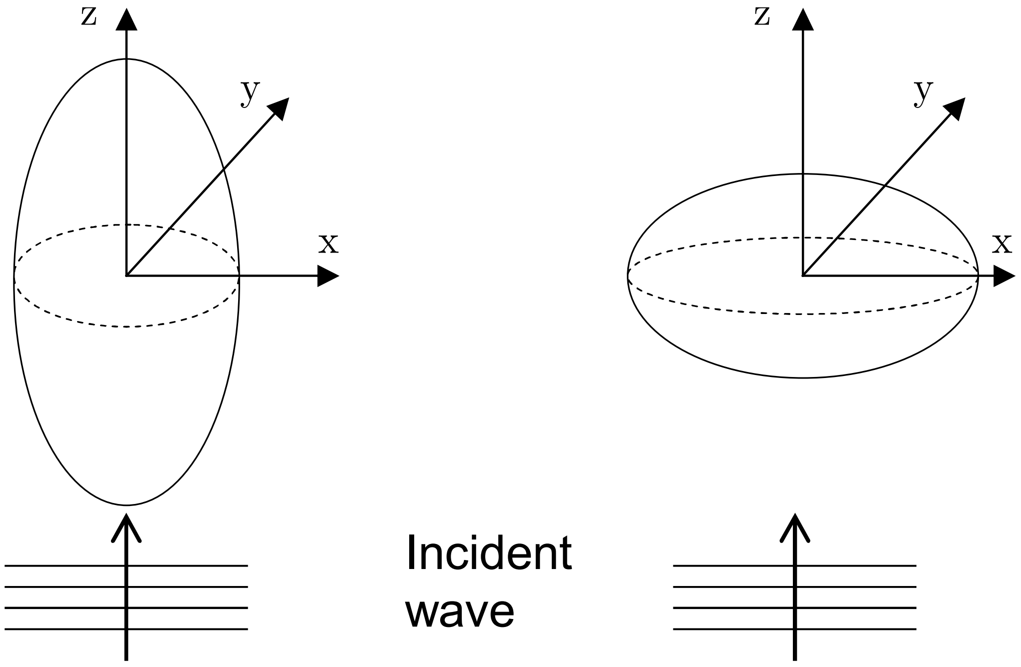
Effective density with spheroids

Does particle shape affect the effective density?

Different shapes ⇒ Different drags

Prolate spheroid: $a_{eq} < a_{pol}$

Oblate spheroid: $a_{eq} > a_{pol}$



Incident wave

↳ Ellipticity: $$e = \sqrt{1 - \left(\frac{\min\left\{ \mathrm{a}_{\mathrm{pol}},\ \mathrm{a}_{\mathrm{eq}} \right\}}{\max\left\{ \mathrm{a}_{\mathrm{pol}},\ \mathrm{a}_{\mathrm{eq}} \right\}} \right)^2}$$

Effective density with spheroids

Ament's model: Effective density

- Conservation of momentum
- Conservation of momentum density
- Drag force = buoyant force

$$\rho_{\text{eff}} = \rho_s + \frac{G}{1-G} c\left(\rho' - \rho\right)$$

$$G := \frac{i\omega\left(1-c\right)\left(\rho'-\rho\right)}{\frac{D}{V} - i\omega\rho}$$

$$D = 6\pi\eta a\left[1 + \beta a - i\beta a\left(1 + \frac{2}{9}\beta a\right)\right]$$

$$\frac{1}{\delta} = \beta = \sqrt{\frac{\rho\omega}{2\eta}}$$

$$\rho_{\text{eff}} = \rho_s - \frac{2c\left(1-c\right)\left(\rho'-\rho\right)^2}{W + iZ}$$

$$\begin{cases} W := 2\left(1-c\right)\left(\rho'-\rho\right) + \frac{9}{2}\frac{1}{\beta a}\rho + 3\rho \\ Z := \frac{9}{2}\left[\frac{1}{\beta a} + \frac{1}{\left(\beta a\right)^2}\right]\rho \end{cases}$$

Effective density with spheroids

Ament's model: Limiting cases

$$\rho_{\text{eff}} = \underbrace{\rho_s - \frac{2c\left(1-c\right)\left(\rho'-\rho\right)^2 W}{W^2+Z^2}}_{\Re\left(\rho_{\text{eff}}\right)} + \underbrace{i\,\frac{2c\left(1-c\right)\left(\rho'-\rho\right)^2 Z}{W^2+Z^2}}_{\Im\left(\rho_{\text{eff}}\right)}$$

Static limit

$$\omega \to 0, \quad \Im\left(\rho_{\text{eff}}\right) \to 0$$

$$\omega \to 0, \quad \Re\left(\rho_{\text{eff}}\right) \to \rho_s$$

$$\omega \to 0, \quad \rho_{\text{eff}} \to \rho_s$$

$$\rho_s = \left(1-c\right)\rho + c\rho'$$

Inviscid limit

$$\omega \to \infty, \quad \Im\left(\rho_{\text{eff}}\right) \to 0$$

$$\omega \to \infty, \quad W \to 2\left(\rho'-\rho\right)\left(1-c\right) + 3\rho$$

$$\omega \to \infty$$

$$\frac{\rho_{\text{eff}}}{\rho} \to \frac{\rho_i}{\rho} = \frac{\left(\rho+2\rho'\right)+c\left(\rho'-\rho\right)}{\left(\rho+2\rho'\right)-2c\left(\rho'-\rho\right)}$$

Effective density with spheroids

Ament's model extension

$$\rho_{\mathrm{eff}} = \rho_s + \frac{G}{1-G} c\left(\rho' - \rho\right)$$

$$G := \frac{i\omega\left(1-c\right)\left(\rho' - \rho\right)}{\dfrac{D}{V} - i\omega\rho}$$

Valid for particle of arbitrary shape

$$D \rightarrow D\left(\mathrm{spheroid}\right), \quad V \rightarrow V\left(\mathrm{spheroid}\right), \quad \rho_{\mathrm{eff}} \rightarrow \rho_{\mathrm{eff}}\left(\mathrm{spheroid}\right)$$

Effective density with spheroids

Drag on a spheroid

Drag on a **prolate** spheroid

[Lai & Mockros, JFM, 1972]

$$D_p = 8\pi\eta a_p \frac{\varepsilon}{\kappa} + \frac{32\pi\eta a_p^2}{3}\frac{\varepsilon^2}{\kappa^2}\sqrt{\frac{\rho\omega}{2\eta}} - i\omega\left[\frac{4}{3}\pi a_{eq}^2 a_p \rho \frac{\left(\frac{1}{\varepsilon^2}-1\right)Q_1\left(\frac{1}{\varepsilon}\right)}{1-\left(\frac{1}{\varepsilon^2}-1\right)Q_1\left(\frac{1}{\varepsilon}\right)} + \frac{32\pi\eta a_p^2}{3}\frac{\varepsilon^2}{\kappa^2}\sqrt{\frac{\rho\omega}{2\eta}}\right]$$

$$\kappa = \frac{1}{2}\left(\frac{1}{\varepsilon^2}+1\right)\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right) - \frac{1}{\varepsilon}$$

$$\varepsilon = \frac{\sqrt{\left(\max\left\{a_{pol}, a_{eq}\right\}\right)^2 - \left(\min\left\{a_{pol}, a_{eq}\right\}\right)^2}}{a_{pol}}$$

$$Q_1(x) = \frac{x}{2}\log\left(\frac{x+1}{x-1}\right) - 1$$

Limiting case: sphere

$$\varepsilon \longrightarrow 0 \qquad D_p \rightarrow 6\pi\eta a\left[1+\beta a - i\beta a\left(1+\frac{2}{9}\beta a\right)\right]$$

Effective density with spheroids

Effective dynamic density with prolate spheroids

$$\rho_{effp} = \rho_s - \frac{(\rho' - \rho)^2 c(1-c) W_p}{(W_p^2 + Z_p^2)} + i\frac{(\rho' - \rho)^2 c(1-c) Z_p}{(W_p^2 + Z_p^2)}$$

$$W_p := \rho\left(\frac{1}{1 - \left(\frac{1}{\varepsilon^2} - 1\right) Q_1\left(\frac{1}{\varepsilon}\right)} + \frac{4}{\beta a_{eq}} \frac{a_{pol}}{a_{eq}} \frac{\varepsilon^2}{\kappa^2}\right) + (\rho' - \rho)(1-c)$$

$$Z_p := \rho\left(\frac{3}{\beta^2 a_{eq}^2} \frac{\varepsilon}{\kappa} + \frac{4}{\beta a_{eq}} \frac{a_{pol}}{a_{eq}} \frac{\varepsilon^2}{\kappa^2}\right)$$

Effective density with spheroids

Limiting cases

Case 1: Ament $a_{pol} \to a_{eq}, \quad \varepsilon \to 0$

$$\rho_{effp} = \rho_s - \frac{(\rho' - \rho)^2 c(1-c) W_p}{(W_p^2 + Z_p^2)} + i \frac{(\rho' - \rho)^2 c(1-c) Z_p}{(W_p^2 + Z_p^2)}$$

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\kappa} = \frac{3}{4}$$

$$\lim_{\varepsilon \to 0} \left[1 - \left(\frac{1}{\varepsilon^2} - 1 \right) Q_1 \left(\frac{1}{\varepsilon} \right) \right] = \frac{2}{3}$$

$$\lim_{\varepsilon \to 0} Z_p = \frac{9}{4} \rho \left(\frac{1}{\beta^2 a^2} + \frac{1}{\beta a} \right) = \frac{Z}{2}$$

$$\lim_{\varepsilon \to 0} W_p = \frac{W}{2}$$

$$\rho_{effp} \to \rho_{eff} = \rho_s - \frac{2c(1-c)(\rho' - \rho)^2 W}{W^2 + Z^2} + i \frac{2c(1-c)(\rho' - \rho)^2 Z}{W^2 + Z^2}$$

Effective density with spheroids

Limiting cases

Case 2: Static $\omega \rightarrow 0, \qquad 1/\beta \rightarrow \infty$

$$\rho_{\text{effp}} = \rho_s - \frac{(\rho' - \rho)^2 c(1-c) W_p}{(W_p^2 + Z_p^2)} + i \frac{(\rho' - \rho)^2 c(1-c) Z_p}{(W_p^2 + Z_p^2)}$$

$$\lim_{(1/\beta)\to\infty} = \frac{(\rho' - \rho)^2 c(1-c)}{W_p + Z_p^2 / W_p} \rightarrow 0 \qquad \lim_{(1/\beta)\to\infty} = \frac{(\rho' - \rho)^2 c(1-c) Z_p}{W_p^2 + Z_p^2} \rightarrow 0$$

$$\rho_{\text{effp}} \rightarrow \rho_s = (1-c)\rho + c\rho'$$

Effective density with spheroids

Limiting cases

Case 3: Inviscid $\omega \to \infty, \quad \beta \to \infty$

$$\rho_{\mathrm{effp}} = \rho_s - \frac{(\rho' - \rho)^2 c(1-c) W_p}{(W_p^2 + Z_p^2)} + i \frac{(\rho' - \rho)^2 c(1-c) Z_p}{(W_p^2 + Z_p^2)}$$

$$\lim_{(1/\beta)\to 0} W_p = \rho E_p + \Delta(1-c)$$

$$\lim_{(1/\beta)\to 0} Z_p = 0$$

$$E_p = \frac{1}{1 - \left(\frac{1}{\varepsilon^2} - 1\right) Q_1\left(\frac{1}{\varepsilon}\right)}$$

$$\frac{\rho_{\mathrm{effp}}}{\rho} = \frac{E_p\left[\rho + c(\rho' - \rho)\right] + (1-c)(\rho' - \rho)}{E_p \rho + (1-c)(\rho' - \rho)}$$

Effective density with spheroids

Effective density with oblate spheroids

Drag on an oblate spheroid

$$D_o = 8\pi\eta a_p \frac{\varepsilon'}{\kappa'} + \frac{32\pi\eta a_p^2}{3}\frac{\varepsilon'^2}{\kappa'^2}\sqrt{\frac{\rho\omega}{2\eta}} - i\omega\left[\frac{4}{3}\pi a_{eq}^2 a_p \rho \frac{\left(\frac{1}{\varepsilon'^2}+1\right) q_1\left(\frac{1}{\varepsilon'}\right)}{1-\left(\frac{1}{\varepsilon'^2}+1\right) q_1\left(\frac{1}{\varepsilon'}\right)} + \frac{32\pi\eta a_p^2}{3}\frac{\varepsilon'^2}{\kappa'^2}\sqrt{\frac{\rho\omega}{2\eta}}\right]$$

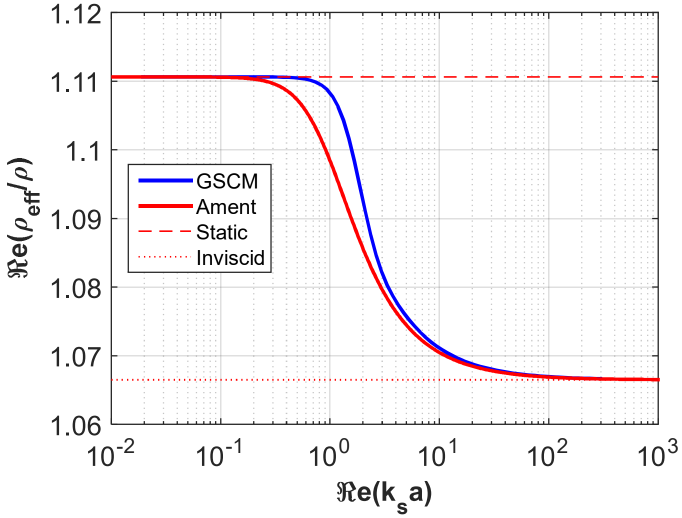
$$\varepsilon' = \sqrt{\frac{a_p^2 - a_{eq}^2}{a_p^2}} = i\varepsilon \qquad q_1(x) = 1 - x\cot^{-1}(x) = -Q_1(ix) \qquad \kappa' = \frac{1}{\varepsilon'} - \left(\frac{1}{\varepsilon'^2} - 1\right)\cot^{-1}\left(\frac{1}{\varepsilon'}\right)$$

$$D_o = 8\pi\eta a_p \frac{\varepsilon}{\kappa} + \frac{32\pi\eta a_p^2}{3}\frac{\varepsilon^2}{\kappa^2}\sqrt{\frac{\rho\omega}{2\eta}} - i\omega\left[\frac{4}{3}\pi a_{eq}^2 a_p \rho \frac{\left(\frac{1}{\varepsilon^2}-1\right) Q_1\left(\frac{1}{\varepsilon}\right)}{1-\left(\frac{1}{\varepsilon^2}-1\right) Q_1\left(\frac{1}{\varepsilon}\right)} + \frac{32\pi\eta a_p^2}{3}\frac{\varepsilon^2}{\kappa^2}\sqrt{\frac{\rho\omega}{2\eta}}\right]$$

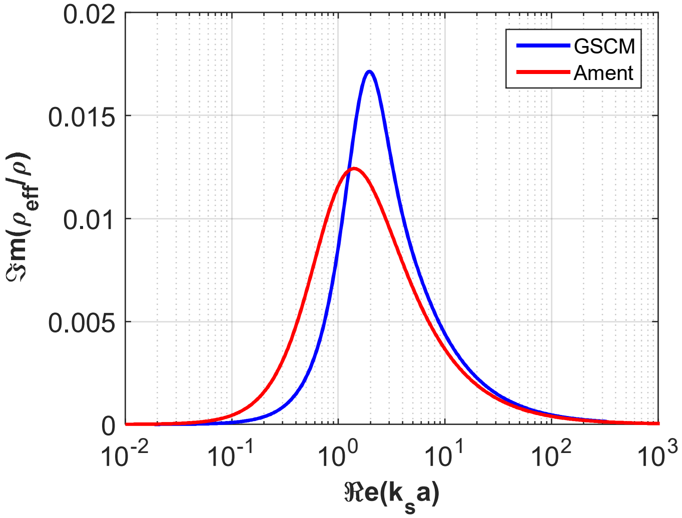
Same drag form ⟶ Same effective density form

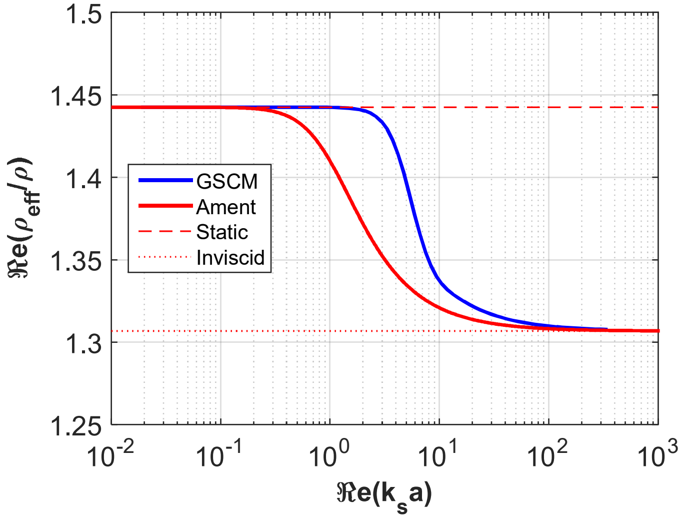
Effective density with spheroids

Numerical results: comparison of Ament model & GSCM

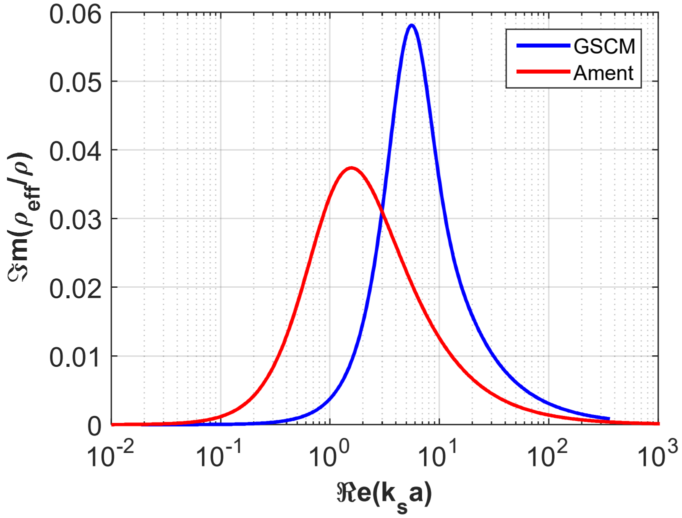


$c = 0.1$



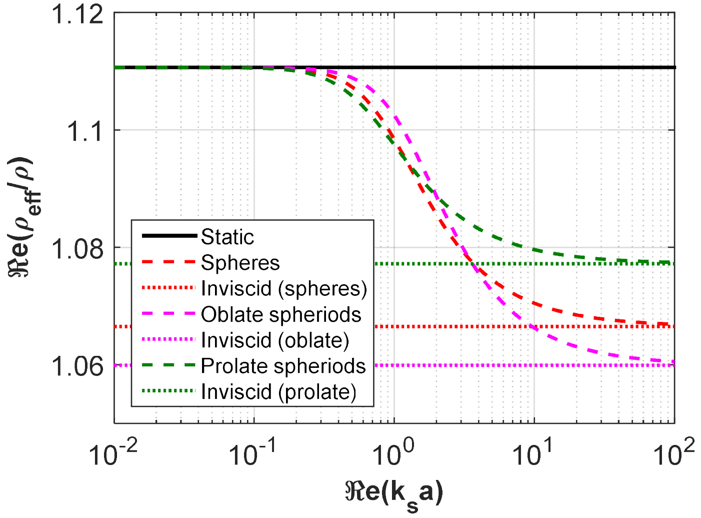


$c = 0.4$

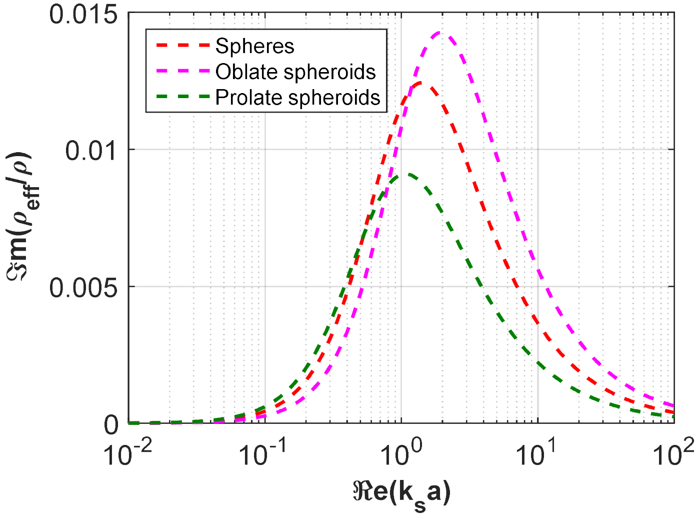


Effective density with spheroids

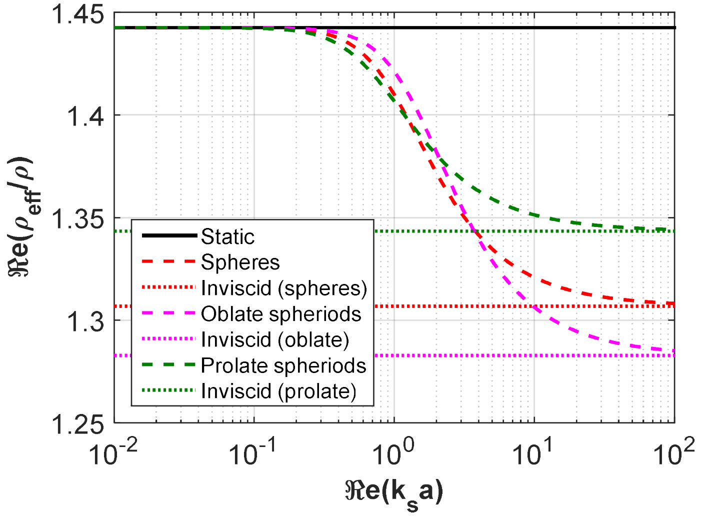
Numerical results: effective density for spheroids and spheres



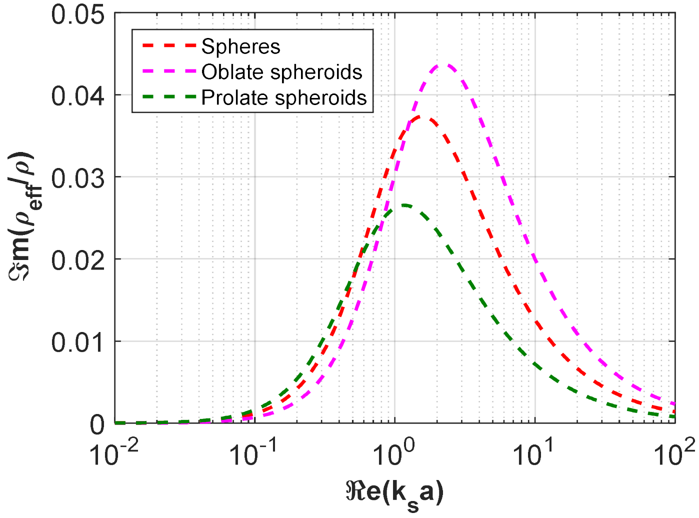
$c = 0.1$



$e = 0.9$



$c = 0.4$



Results

- Calcuation of the effective mass density for a suspension of both prolate spheriods and oblate spheriods by extending an hydrodynamic model from spherical particles to spheriodal ones.

Outline

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1. Motivation
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5. Effective Properties with Spheroids
6. Conclusion & Summary
7. Perspectives

Conclusion & Summary

- Analytical expressions for scattering coefficients for arbitrary partial wave orders for a sphere in a viscous fluid.

- Extension of the core-shell effective medium through inclusion of the fluid viscosity and determination of effective properties of using the model.

- Investigation of the effective dynamic density with spheroidal particles by extending a hydrodynamic model.

Future work

- Calculating scattering coefficients for a spheroid taking wave mode conversion into consideration
- Calculating scattering coefficients for a core-shell particle.
- Calculating the effective shear/compressional wavenumber using the core-shell effective medium model

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cnrs dépasser les frontières

Thank you

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Publications

Loughborough University

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Laboratoire Ondes Milieux Complexes

Alam, M. Mahbub, et al. "Effective dynamic properties of random complex media with spherical particles." *The Journal of the Acoustical Society of America* 145.6 (2019): 3727-3740.

Alam, Mahbub, et al. "The coherent shear wave in suspensions." *Journal of Physics: Conference Series*. Vol. 1017. No. 1. IOP Publishing, 2018.

Alam, M. Mahbub, et al. “Scattering coefficients for a sphere for a visco-acoustic media for arbitrary partial wave order" *Wave Motion* (2020): 102589

Alam, M. M. (2019). *Acoustic wave propagation through a random dispersion of solid particles in a viscous fluid* (Doctoral dissertation, Normandie Université).

Alam, M. Mahbub, et al. "Effective dynamic properties of random complex media." *The Journal of the Acoustical Society of America* 144.3 (2018): 1961-1961.

Alam, M. Mahbub. "Ultrasonic propagation in concentrated colloidal dispersions: improvements in a hydrodynamic model." *Journal of Dispersion Science and Technology* 43.8 (2022): 1177-1186

Motivation

Why be interested in effective properties ?

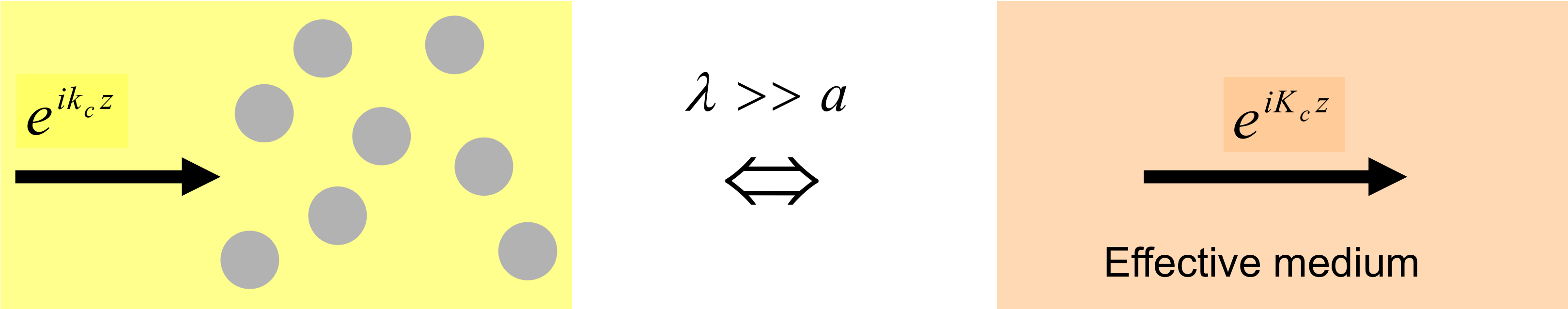
- Determine the concentration of particles.
  For example, the encapsulated medics in an organ.

- Predict average propagation in a complex medium
  and interaction with a defect as a tumor in an organ.

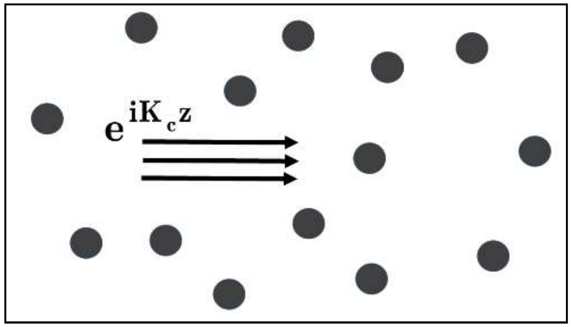
+ Solve interface / Scattering problems

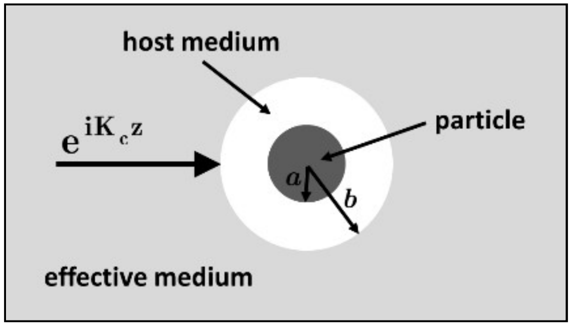
Effective density with spheres

Effective medium



Core-shell model



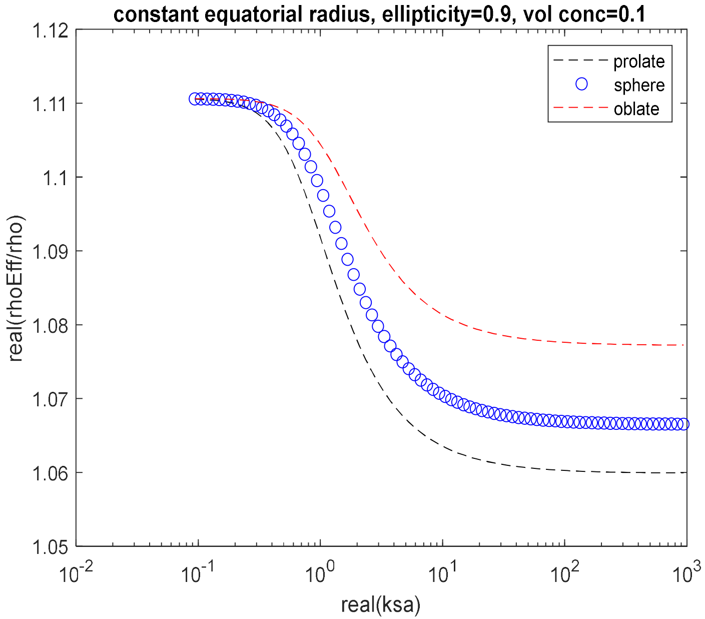


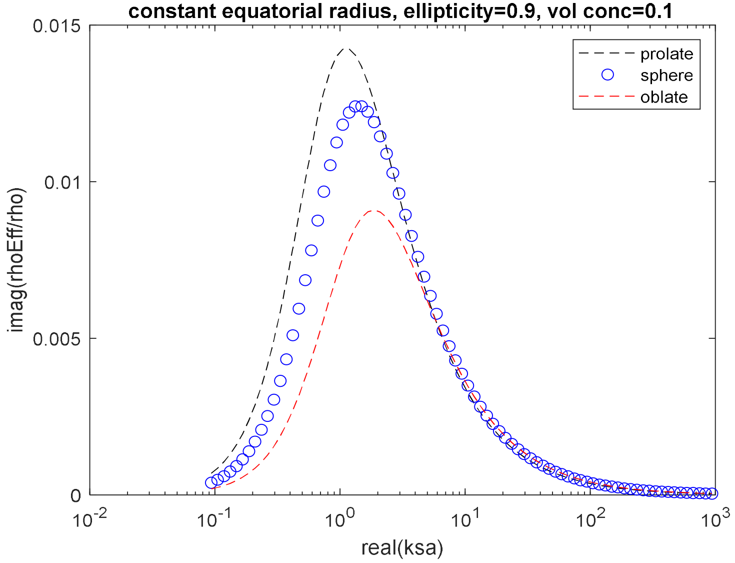
Concentration $c = \dfrac{a^3}{b^3}$

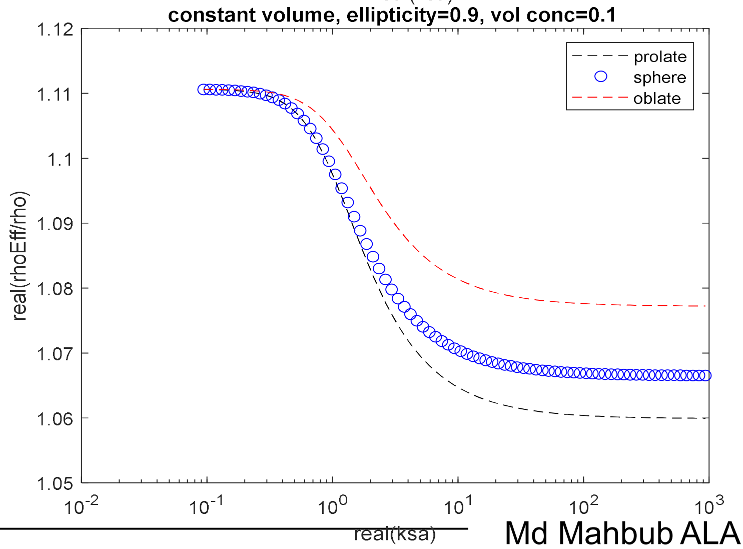
Self-consistency condition $t_n^{pq} = 0$

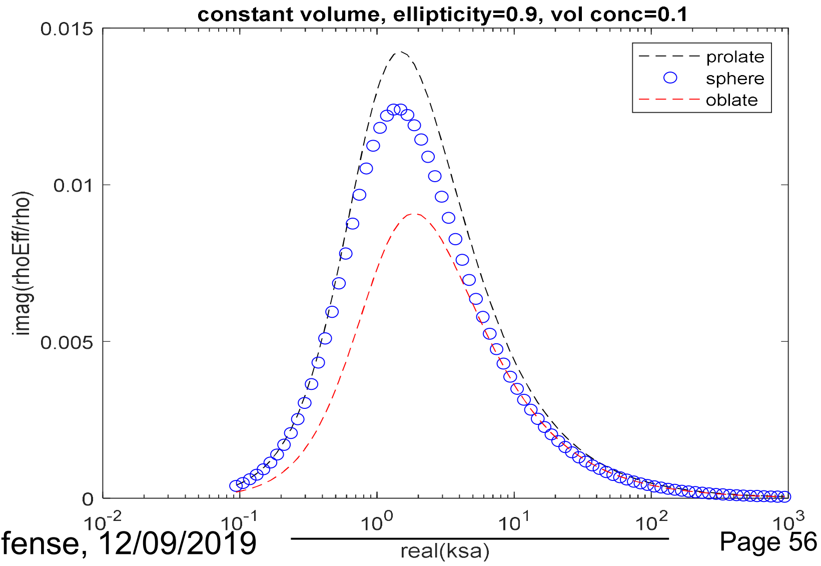
Effective density with spheroids

Effective density with spheroids


constant equatorial radius, ellipticity=0.9, vol conc=0.1


constant equatorial radius, ellipticity=0.9, vol conc=0.1


constant volume, ellipticity=0.9, vol conc=0.1


constant volume, ellipticity=0.9, vol conc=0.1